Substitution

Type 6) $\dfrac{dy(t)}{dt} = F\left(y/t\right)$

$\uparrow$ arbitrary expression $\sin(y/t)$, $(y/t)^3$, ...

Substitution: $u = y/t \Rightarrow y = u \cdot t$

take derivative w.r.t. t:

$$\frac{dy}{dt} = u + \frac{du}{dt} \cdot t$$

plug in ODE:

$$F(u) = u + \frac{du}{dt} \cdot t$$

$$(F(u) - u) \cdot dt = du \cdot t$$

↳ this ODE is seperable

$$\frac{1}{t}\, dt = \frac{1}{(F(u) - u)} \cdot du$$

$$\underbrace{\int_{t_0}^{t} \frac{1}{\tilde{t}}\, d\tilde{t}}_{\ln\left(\frac{t}{t_0}\right)} = \underbrace{\int_{u_0}^{u} \frac{1}{F(\tilde{u}) - \tilde{u}}\, d\tilde{u}}_{\text{depends on } F}$$

Example:

$$\frac{dy}{dx} = \frac{y}{x} + 1$$

$$= \underbrace{\frac{1}{x}}_{g(x)} \cdot \underbrace{y}_{h(y)} + \underbrace{1}_{\text{perturbation}}$$

linear

- dependent variable y
- independent " x
- linear as y comes to 1^{st} order
- inhomogeneous, as there is a perturbation function $(\equiv 1)$

Substitution: $u = \frac{y}{x}$

ODE: $y' = u + 1 = F(u)$

Here: $\ln\left(\frac{x}{x_0}\right) = \int_{u_0}^{u} \frac{1}{\tilde{u} + 1 - \tilde{u}}\, d\tilde{u}$

$$= u - u_0 \quad | \; e^{\cdots}$$

$$\left[\quad \frac{x}{x_0} = e^{u - u_0} = e^{y/x - \frac{y_0}{x_0}} \quad \right]$$

backsubstitution

redundant

Solve for $y(x)$:

$$\ln\left(\frac{x}{x_0}\right) = \frac{y}{x} - \frac{y_0}{x_0}$$

$$y(x) = x \ln\left(\frac{x}{x_0}\right) + \frac{y_0}{x_0} \cdot x$$

Existence and Uniqueness of solutions

Example: $\frac{dy}{dx} = \frac{1-y}{x}$

Solution by separation of variables

$$\frac{1}{1-y}\,dy = \frac{1}{x}\,dx$$

$$\underbrace{-\int_{y_0}^{y} \frac{-1}{1-\tilde{y}}\,d\tilde{y}}_{\text{log. integral of type } \frac{f'(\tilde{x})}{f(\tilde{x})}} = \int_{x_0}^{x} \frac{1}{\tilde{x}}\,d\tilde{x}$$

$$-\underbrace{\ln\left|\frac{1-y}{1-y_0}\right|} = \ln\left|\frac{x}{x_0}\right|$$

$$\ln\left|\frac{1-y_0}{1-y}\right| = \ln\left|\frac{x}{x_0}\right| \quad | \; e^{\cdots}$$

$$\frac{1-y_0}{1-y} = \frac{x}{x_0}$$

in general 4 cases: $\pm, \pm$ for both $|\ldots|$

$$\frac{1-y}{1-y_0} = \frac{x_0}{x}$$

$$1-y = \frac{x_0}{x}\cdot(1-y_0)$$

- no solution at $x = 0$

$$y = 1 - \frac{x_0(1-y_0)}{x}$$

Example 2): $\frac{dy}{dx} = \frac{y-1}{x}$

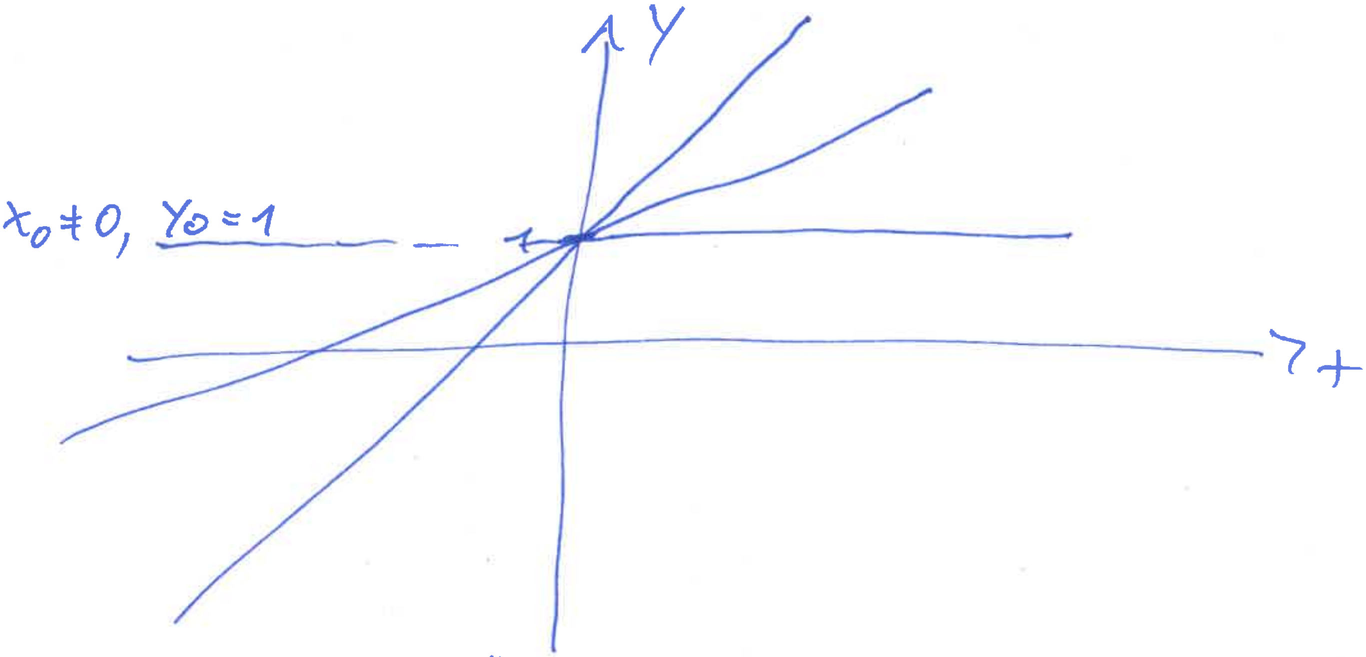
similar steps up to a - sign

$$\ln\left|\frac{y-1}{y_0-1}\right| = \ln\left|\frac{x}{x_0}\right| \quad | \, e^{\cdots}$$

$$y - 1 = (y_0 - 1)\cdot\frac{x}{x_0}$$

$$y(x) = 1 + \frac{y_0-1}{x_0}\cdot x$$

plot the solutions:



Solutions differ due to (x_0, y_0)

- all points of $(x-y)$-plane can be reached except $(x=0, y\neq 1)$ (for $(x=0, y\neq 1)$ no solution exists)
- For all points (x, y) except $(x=0, y=1)$ the solution is unique

Solution method for <u>exact</u> <u>ODE</u>

Review: Definition of the total derivative of a function $U(x,y)$ of two variables:

$$dU = \boxed{\frac{\partial U(x,y)}{\partial x}} \cdot dx + \frac{\partial U(x,y)}{\partial y} \cdot dy$$

<u>Def</u>: Exact ODE

A first order first degree ODE is called exact if

$$\boxed{A(x,y)}\, dx + B(x,y)\, dy = 0$$

and $\dfrac{\partial A(x,y)}{\partial y} = \dfrac{\partial B(x,y)}{\partial x}$

If an ODE is exact then it represents the total differential of a potential function $U(x,y)$

- application in thermodynamics

$$dU(x,y) = 0$$

$$A(x,y) = \frac{\partial U(x,y)}{\partial x}, \quad B(x,y) = \frac{\partial U(x,y)}{\partial y}$$

Exactness: $\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$

plug in what A and B are:

$$\frac{\partial^2 U(x,y)}{\partial y\,\partial x} = \frac{\partial^2 U(x,y)}{\partial x\,\partial y}$$

Exactness means that you can change the order of taking derivatives w.r.t. x or y

<u>Solution method:</u>

(1) Check if ODE is exact

(2) Construct the potential function $U(x,y)$:

(2.1) $dU(x,y) = 0$

$\Rightarrow U(x,y) = C_1$ (constant)

(2.2) Use $\boxed{A(x,y) = \frac{\partial U(x,y)}{\partial x}}$

$$\underbrace{U(x,y)}_{\text{intermediate result for } U} = \int A(x,y)\,dx + \underbrace{F(y)}_{y\text{-dependent constant}}$$

(2.3) Plug the intermediate result into

$$B(t, y) = \frac{\partial U(t, y)}{\partial y}$$

i.e. calculate the partial derivative of the intermediate result w.r.t y

(2.4) This generates a new ODE for $F(y)$

- solve it and
- plug the solution for $F(y)$ into the intermediate result for $U(t, y)$

(2.5) Solve for y in the expression of $U(t, y) = C_1$

<u>Example</u>: $2t\dfrac{dy}{dt} + 5t + 2y = 0$

(1) Check if ODE is exact:

(1.1). Re-write ODE in A-B-form:

$$2t\,dy + (5t+2y)\,dt = 0$$

$$\underbrace{(5t+2y)}_{A(t,y)}dt + \underbrace{2t}_{B(t,y)}dy = 0$$

(1.2) Check definition:

$$\frac{\partial A(t,y)}{\partial y} = \frac{\partial B(t,y)}{\partial t}$$

$$2 \overset{?}{=} 2 \quad \text{ok} \checkmark$$

yes, ODE is exact

(2) Construct potential function:

(2.1) $U(t,y) = C_1$ (always true)

(2.2) Integrate $A(t,y) = \dfrac{\partial U(t,y)}{\partial t}$

$$U(t,y) = \int A(t,y)\,dt + F(y)$$

$$= \int (5t+2y)\,dt + F(y)$$

$$= \frac{5}{2}t^2 + 2yt + F(y)$$

(intermediate result)

(2.3) Plug intermediate result into $B(t,y) = \dfrac{\partial U(t,y)}{\partial y}$

$$2t = \frac{\partial}{\partial y}\left[\frac{5}{2}t^2 + 2yt + F(y)\right]$$

$$2t = 2t + \frac{dF(y)}{dy}$$

we obtain an ODE for $F(y)$:

$$\frac{dF(y)}{dy} = 0 \Rightarrow F(y) = C_2 \text{ constant}$$

(2.4) Insert $F(y) = C_2$ into intermediate result:

$$U(t,y) = \frac{5}{2}t^2 + 2yt + C_2 = \boxed{C_1}$$

(2.5) Extract solution $y(t)$ from

$$\frac{5}{2}t^2 + 2yt + C_2 = C_1$$

$$y(t) = \frac{C_1 - C_2}{2t} - \frac{5}{4}t$$

(solution)

- Fit constant $C = C_1 - C_2$ by using initial conditions

$$y_0 = \frac{C}{2t_0} - \frac{5}{4}t_0 \Rightarrow C = \ldots$$

Test the solution:

ODE $\quad 2x\,y' + 5x + 2y = 0$

Calculate $\quad y'(x) = -\frac{c}{2x^2} - \frac{5}{4}$

$$2x\left(\frac{-c}{2x^2} - \frac{5}{4}\right) + 5x + 2\left(\frac{c}{2x} - \frac{5}{4}x\right)$$

$$= \frac{-c}{x} - \frac{5}{2}x + 5x + \frac{c}{x} - \frac{5}{2}x$$

$$= 0 \quad \text{ok}$$