B Solving first order ODEs

B 1. Solving seperable, homogeneous first order ODEs

General notation:

$$\frac{d\, f(x)}{d\boxed{x}} = F(x, f(x))$$

$\boxed{x}$ independent variable

$f(x)$ dependent "

Alternative notation:

$$A(x, f(x))\,dx + B(x, f(x))\,dy = 0$$

Check: $\displaystyle \frac{dy}{dx} = - \frac{A(x, f(x))}{B(x, f(x))} =: F$

$y = f(x)$

if the ODE is separable

$$\boxed{\frac{df(x)}{dx} = g(x) \cdot h(\underbrace{f(x)}_{y})}$$

product of two terms

Solution method:
"separation of variables"

Starting point: $\frac{df(t)}{dt} = g(t) \cdot h(f(t))$

(1) Separation of t and $f(t)$:

$$\frac{d\boxed{f(t)}}{h(f(t))} = g(t)\, dt$$

(2) Integrate both sides:

- initial condition $f(t_0)$ at t_0
- integrate to an arbitrary point $(t, f(t))$

$$\int_{f(t_0)}^{f(t)} \frac{1}{h(\tilde{f}(t))}\, d\tilde{f}(t) = \int_{t_0}^{t} g(\tilde{t})\, d\tilde{t}$$

(3) Solve for $f(t) = \ldots$

Examples:

1) $\frac{dP(t)}{dt} = \lambda P(t)$

- first order ✓
- dependent variable $P(t)$
- independent variable t

- $g(t) = \lambda$ (constant)
- $h(P(t)) = P(t)$

Solution method:

(1) $$\frac{dP(t)}{P(t)} = \lambda \cdot dt$$

(2) Integrate on both sides

$$\int_{P(t_0)}^{P(t)} \frac{1}{\tilde{P}(t)}\, d\tilde{P}(t) = \int_{t_0}^{t} \lambda\, d\tilde{t}$$

$$\left[\ln |\tilde{P}(t)| \right]_{P(t_0)}^{P(t)} = \lambda \left[t \right]_{t_0}^{t}$$

$$\ln |P(t)| - \ln |P(t_0)| = \lambda (t - t_0)$$

$$\ln \left| \frac{P(t)}{P(t_0)} \right| = \lambda (t - t_0) \quad \Big| \; e^{\cdots}$$

use $P(t) \geq 0$ for all t

[in general two cases!]

$$\frac{P(t)}{P(t_0)} = e^{\lambda (t - t_0)}$$

(3) Solve for $P(t)$

$$P(t) = P(t_0) \cdot e^{\lambda (t - t_0)}$$

2) $\dfrac{dy(t)}{dt} = t \cdot y(t)$ with $y(t) > 0$

(1) Separation of variables

$$\frac{dy(t)}{y(t)} = t\,dt$$

(2) Integrate both sides

$$\int_{y(t_0)}^{y(t)} \frac{d\tilde{y}(t)}{\tilde{y}(t)} = \int_{t_0}^{t} \tilde{t}\,d\tilde{t}$$

$$\underbrace{\ln|y(t)| - \ln|y(t_0)|}_{\ln\left|\frac{y(t)}{y(t_0)}\right|} = \frac{1}{2}(t^2 - t_0^2)$$

$$\frac{y(t)}{y(t_0)} = e^{1/2(t^2 - t_0^2)}$$

(3) $y(t) = y(t_0)\, e^{1/2(t^2 - t_0^2)}$

3) $\frac{dy}{dx} = \sqrt{y(t)} = y(t)^{1/2}$

• initial condition $y(0) = 0$

Educated guess:

$y(t) = Ax^2$

$\frac{dy}{dx} = 2Ax$ and $\sqrt{y} = \sqrt{A} \cdot |x|$

is $2Ax = \sqrt{A}\,|x|$

requirement $2A = \sqrt{A}$

(up to sign...) $4A^2 = A$

$4A^2 - A = A(4A - 1) = 0$

$\hookrightarrow$ $A_1 = 0$

$A_2 = 1/4$

we have found two solutions:

$Y_1(t) = 0$ everywhere

$Y_2(t) = \frac{1}{4}x^2$ for positive x

Now: Use solution method

(1) separation of variables:

$$\frac{dy}{y^{1/2}} = dx$$

(2) Integrate both sides:

$$\int_{y_0}^{y} \frac{d\tilde{y}}{\tilde{y}^{1/2}} = \int_{t_0}^{t} d\tilde{t}$$

$$\left[2\tilde{y}^{1/2}\right]_{y_0}^{y} = \left[\tilde{t}\right]_{t_0}^{t}$$

$$2\sqrt{y} - 2\sqrt{y_0} = t - t_0 \quad \Big| :2 \Big| + \sqrt{y_0}$$

(3) Solve for $y(t)$

$$\sqrt{y} = \frac{t - t_0}{2} + \sqrt{y_0} \quad \Big|^2$$

$$y(t) = \left(\frac{t - t_0}{2} + \sqrt{y_0}\right)^2$$

Compare with educated guess:

Case $y_2(t) = \frac{1}{4}t^2$ is special case of $y(t)$ for $(t_0, y_0) = (0, 0)$

↳ solution method produced a more general solution

↳ but solution method has not found all solutions (y_1 not found)

Substitution

↳ general approach
↳ specify some types:

a) Type: $y' = f(ax + by + c)$

- dependent variable $y(x)$
- independent " x
- f arbitrary expression

(1) Substitution: $u = ax + by + c$

(2) Calculate differential

$$\frac{du}{dx} = \frac{\partial u(x)}{\partial x} + \frac{\partial u}{\partial y(x)} \cdot \frac{\partial y(x)}{\partial x}$$

$$= a + b \cdot \underbrace{\frac{\partial y(x)}{\partial x}}_{\text{left hand side of ODE}}$$

$$= a + b \cdot f(u)$$

$$= \underbrace{1}_{} \cdot \underbrace{(a + b \cdot f(u))}_{\text{only depends on } u}$$

↳ separation of variables is possible here!

(3) Use a suitable solution method (here: separation of variables)

(1) $$\frac{du}{a+b\cdot f(u)} = dx$$

(2) Integrate both sides

$$\int_{u_0}^{u} \frac{1}{a+b\cdot f(\tilde{u})}\, d\tilde{u} = \int_{x_0}^{x} d\tilde{x}$$

now it depends on the details of $f(u)$!

(3) Solve for variable $u = \ldots$ (*)

(4) Plug in definition of u

$$ax + b\,y(x) + c = (*)$$

and solve for $y(x)$

<u>Example:</u> $$\frac{dy}{dx} = x + y(x)$$

- first order, first degree
- linear equation: $y(x)$ is first order
- inhomogeneous equation perturbation function $g(x) = x$

Check: Substitution with type a)

$$\frac{dy}{dt} = f(at + by + c)$$

- $f = \mathrm{id} \iff f(t) = t$
- $a = 1,\ b = 1,\ c = 0$

$\rightarrow$ go to step (3) c2):

$$\int_{u_0}^{u} \frac{1}{1+\tilde{u}}\, d\tilde{u} = \int_{t_0}^{t} dt$$

Recap: $\int \frac{1}{t}\, dt = \ln|t| + C$

$\int \frac{f'(t)}{f(t)}\, dt = \ln|f(t)| + C$

Here: $f(\tilde{u}) = 1 + \tilde{u},\quad f'(\tilde{u}) = 1$

$$\left[\ln|1+\tilde{u}|\right]_{u_0}^{u} = [t]_{t_0}^{t}$$

$$\ln\left|\frac{1+u}{1+u_0}\right| = t - t_0 \quad | \; e^{\cdots}$$

assume $u \geq 0$ for all t

$$\frac{1+u}{1+u_0} = e^{t-t_0}$$

(3) Solve for variable:

$$v(t) = (1+v_0)\cdot e^{t-t_0} - 1$$

(4) Plug in definition of v

$$v = at + by + c = t + y(t)$$

$$t + y(t) = (1 + (t_0 + y_0(t_0))e^{t-t_0} - 1$$

$$\Rightarrow y(t) = (1 + (t_0 + y_0))e^{t-t_0} - t - 1$$

solution for original ODE!