Ordinary Differential Equations (ODEs)

Getting started:

Finite difference equations

Example:

$t=0, x_0, v_0$

$t=1, x_1, v_1$

$t=2, x_2, v_2$

change in position Δx
" " time Δt $\Big\} \to v = \frac{\Delta x}{\Delta t}$
" " velocity Δv

physical observation $\frac{\Delta v}{\Delta t} = g$
("law of nature")

Mathematics: $\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} =: \frac{dv}{dt}$

Reconstruction of full time trajectory: $x(t)$
↳ integration over ODE

<u>Def</u>: A differential equation is an equation which contains (one or more) derivatives ($\frac{d}{dx_i}$) of an unknown function $f(\vec{x})$ with respect to (w.r.t.) the <u>independent</u> variables $\vec{x} = (x_1, x_2, \dots x_n)$

If, furthermore,

(i) all derivatives are w.r.t. a single variable only it is called an <u>ordinary</u> differential equation

(ii) derivatives w.r.t. several variables occur it is called a <u>partial</u> differential equation (PDE)

<u>Example</u>:

(i) $f'(x) = -3f(x)$ only variable $x \in \mathbb{R}$

(ii) $$\frac{\partial T(x,t)}{\partial t} = \frac{\lambda}{c\rho} \frac{\partial^2 T(x,t)}{\partial x^2}$$

λ: heat conductivity, c: specific heat
T: temperature field depends on x <u>and</u> t

Classification of ODEs:

(1) Order: An ODE is called of order n if the highest derivative involved is the n-th derivative

(2) degree: the degree is the power (highest) to which the highest derivative appears

(3) linear: An ODE is linear if the unknown function $f(t)=y$ (also called dependent variable $y=y(t)$) appears to first order only e.g. $\frac{df(t)}{dt} = f(t)$

here: f has order 1

nonlinear: e.g. $\frac{df(t)}{dt} = f^3(t)$

here: f has order 3

but: order of ODE is still 1 because highest derivative is first derivative!

(4) Explicit: An ODE of order n is called explicit if the highest order derivative can be extracted:

$$\frac{d^n f(t)}{dt^n} = F\left(x, f(t), \frac{df}{dt}, \frac{d^2 f}{dt^2}, \dots \frac{d^{n-1} f}{dt^{n-1}}\right)$$

Example: $\exp\left(\frac{d^n f(t)}{dt^n}\right) + \log\left(\frac{d^n f(t)}{dt^n}\right)$

$= \dots$

can never be explicit!

Implicit: if an ODE is not explicit, it is implicit

(5) Autonomous (dt. autonom):

An explicit ODE is called autonomous if $F(\dots)$ does not depend on the independent variable x directly)

(6) <u>Homogeneous:</u> An ODE is called homogeneous, if all its terms depend on the dependent variable $f(t)$ or of its derivatives

↳ equivalent: the ODE does not contain a <u>perturbation</u> (dt. Störfunktion)

<u>Example:</u> $\frac{d f(t)}{dt} = \alpha(t) f(t) = F(\dots)$

↳ not autonomous!
$\alpha(t)$ appears in $F(\dots)$ and introduces a direct dependence on t

↳ but it is homogeneous because term $\alpha(t) f(t)$ contains $f(t)$ as a factor

<u>Inhomogeneous:</u>

$$\frac{d f(t)}{dt} = f(t) + \underbrace{g(t)}_{\text{perturbation}}$$

$g \neq f$

(7) Seperable: An ODE is called seperable if it can be written as a product of two functional expressions of the
- independent variable t
- and of the dependent variable $f(t)$

e.g. $$\frac{df(t)}{dt} = g(t) \cdot h(f(t))$$

Def: Initial value problem (IVP) is defined by
- an ODE of order n
- a set of n initial conditions $f^{(i)}(t_0) = y_0^i$

Def: Boundary value problem
- an ODE of order n
- set of n boundary values $(t_i, f(t_i)) = (t_i, y_i)$

(A) General methods for solving ODEs

(1) Educated guess:

- unsystematic

Example: $y''(t) + y(t) - t = 0$

Classification:

- order 2
- degree 1
- linear yes, $y(t)$ is first order
- explicit yes, $y'' = t - y(t) = F(t, y(t))$
- autonomous: no, because t in $F(t, \dots)$
- homogeneous: perturbation $g(t) = -t$ no
- seperable: no, no product

Guess: $y(t) = t$, $y'(t) = 1$, $y''(t) = 0$ } tested by inserting into ODE

$0 + t - t = 0$ ✓ ok

solution has been found, however, there could be other solutions (→ later)

(2) Substitution

$$\frac{dy}{dx} = f(h(x,y)) \quad \text{(ODE)}$$

say $u = h(x,y)$ is substitution

represent ODE using u

need
$$\frac{du}{dx} = \frac{\partial h(x,y)}{\partial x} + \frac{\partial h(x,y)}{\partial y} \cdot \underbrace{\frac{\partial y}{\partial x}}_{\text{from ODE}}$$

$$= \frac{\partial h(x,y)}{\partial x} + \frac{\partial h(x,y)}{\partial y} \cdot f(h)$$

Example: