Solution method for linear, inhomogeneous first order ODEs

Recap: Linearity

Let $f(x)$ be an arbitrary function

$$y_1 = f(x_1) \qquad x_1, x_2 : \text{two arbitrary points}$$
$$y_2 = f(x_2)$$

$$y_1 + y_2 = f(x_1) + f(x_2) \quad \text{(always true)}$$
$$\underset{\text{linear}}{\overset{\text{if } f \text{ is}}{=}} f(x_1 + x_2)$$

e.g. $f(x) = x^2$

$$y_1 + y_2 = x_1^2 + x_2^2 \neq (x_1 + x_2)^2$$

e.g. $f(x) = ax$

$$y_1 + y_2 = ax_1 + ax_2 = a(x_1 + x_2)$$

Def: A linear inhomogeneous first order ODE is of the following form:

$$\frac{dy(x)}{dx} + p(x)\underset{\text{linear}}{y} = \underbrace{q(x)}_{\text{perturbation}}$$

$$\Leftrightarrow \underset{B(x,y)}{1}\, dy(x) + \underbrace{\left(p(x)\,y - q(x)\right)}_{A(x,y)} \cdot dx = 0$$

Solution: Finding integrating factor

↳ as ODE is linear the integrating factor $\mu(x)$ depends on x only

↳ multiply ODE with $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + \mu(x)\,p(x)\cdot y = \mu(x)\cdot q(x)$$

write this as a total derivative

$$\frac{d}{dx}\left[\mu(x)\,y\right] = \frac{d\mu}{dx}\cdot y + \mu(x)\cdot\frac{dy}{dx}$$

this is possible if the --- -terms agree:

(I) $\dfrac{d\mu}{dx} = \mu(x)\cdot p(x)$ (determines μ)

(II) $\dfrac{d}{dx}\left[\mu(x)\,y\right] = \mu(x)\,q(x)$ (needs to be solved for result)

Solution:

(1) Calculate $\mu(x)$ from eqn (I):

$$\int\frac{d\mu(x)}{\mu(x)} = \int p(x)\,dx$$

$$\ln(\mu(x)) = \int p(x)\,dx \quad | \, e^{\cdots}$$

$$\boxed{\mu(x) = e^{\int p(x)\,dx}} \quad (1)$$

(2) Solve the ODE in (II)

$$d[\mu(t)\, y] = \mu(t)\, q(t)\, dt \quad |\int$$

$$\mu(t) \cdot y = \int \mu(t)\, q(t)\, dt$$

$$\boxed{y = \frac{1}{\mu(t)} \cdot \int \mu(t)\, q(t)\, dt} \quad (2)$$

This method is also known in books as the "variation of constants"

Example: $\frac{dy}{dt} + \underbrace{2t}_{p(t)} y = \underbrace{4t}_{q(t)}$

(1) Integrating factor:

$$\mu(t) = e^{\int p(t)\, dt} = e^{\int 2t\, dt} =$$

$$= e^{t^2}$$ (integration constant not relevant for $\mu(t)$)

(2) Integrate

$$\underline{y} = e^{-t^2} \int e^{t^2} \cdot 4t\, dt = 2e^{-t^2} \int \underbrace{2t e^{t^2}}\, dt$$

$$= 2e^{-t^2} \cdot (e^{t^2} + c_1) = \underline{2 + 2c_1 \cdot e^{-t^2}}$$

General solution for linear inhomogeneous first order ODEs:

Theorem: The solution

$y(t) = y_h(t) + y_p(t)$

is given by a superposition of the general solution of the corresponding homogeneous ODE (called y_h) and a particular solution of the inhomogeneous ODE (called y_p)

Solution method:

(1) Solve corresponding homogeneous ODE (i.e. for $\varphi(t) = 0$)

(2) Find a particular solution of the inhomogeneous ODE, e.g. by calculating the integrating factor

(2) Find particular solution for the inhomogeneous ODE:

(2.1) Find the integrating factor

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int -\frac{1}{t}\,dt}$$

$$= e^{-\ln(t)} = e^{\ln(t^{-1})} = \frac{1}{t}$$

(2.2) Integrate to find y_p:

$$y_p(t) = \frac{1}{\mu(t)} \cdot \int \mu(t) \cdot q(t)\,dt$$

$$= t \cdot \int \frac{1}{t} \cdot 5t\,dt = t \cdot 5t = 5t^2$$

$$\boxed{y_p(t) = 5t^2}$$

(3) General solution $y(t) = y_h + y_p$

$$y(t) = y_0 \cdot t + 5t^2$$

(4) Solve the initial value problem (if required)

$$y(1) = 0 = y_0 \cdot 1 + 5 \cdot (1)^2 = y_0 + 5$$

$$\Rightarrow y_0 = -5$$

$$y_{IVP}(t) = -5t + 5t^2$$

Why are there many solutions?

ODE: $\frac{dy_{inh}}{dt} + p(t)\, y_{inh} = q(t)$ (inhom.)

$$\frac{dy_h}{dt} + p(t)\cdot y_h = 0 \quad \text{(homog.)}$$

$$\frac{dy_{inh}}{d\cancel{x}} + \frac{dy_h}{dt} + p(t)\, y_{inh} + p(t)\, y_h = q(t)$$

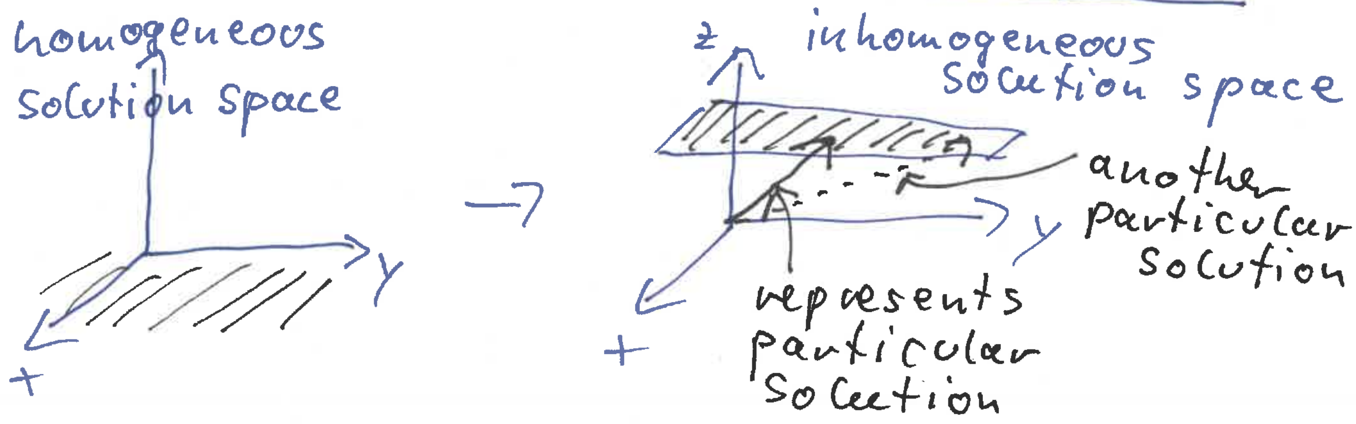
$$\frac{d}{dt}\left[y_{inh} + y_h\right] + p(t)\left[y_{inh} + y_h\right] = q(t)$$

So the combination

$y_{inh} + y_h$ is always a solution

of the inhomogeneous ODE!

Visualization of solution space



Example: Assume initial value problem (IVP)

$y(1) = 0$

$$\frac{dy}{dx} = \frac{y}{x} + 5x .$$

(0) Compare with standard form

$$\frac{dy}{dx} \underbrace{- \frac{1}{x}}_{p(x)} \cdot y = \underbrace{5x}_{q(x)}$$

(1) Solve the corresponding homogeneous ODE: $\rightarrow$ set $q(x) = 0$

$$\frac{dy}{dx} = \frac{1}{x} \cdot y \quad | \cdot dx : y$$ separation of variables

$$\frac{dy}{y} = \frac{dx}{x}$$ include one constant in calculation for first order ODE

$$\ln(y) - \ln(y_0) = \ln(x)$$

$$\ln(y/y_0) = \ln(x) \quad | e^{\cdots}$$

$$y_h(x) = y_0 \cdot x$$ (homogeneous solution!)