Kapitel 1

Ordinary differential equations (ODE)

1.1 From finite difference to differential equations

Laws of nature and many similar theories in other fields of research analyze the change of functions w.r.t. variables. Those changes are studied for small examples and then continued to larger situations. This approach naturally leads to the concept of a finite difference equations.

Example: How changes the trajectory of a vertically falling stone between two observations?
We perform a measurement of position and velocity at a first $(t_1; x_1, v_1)$ and a second $(t_2; x_2, v_2)$ point in time and can find (approximate) relations between them. This allows to derive finite difference relations of the form

$$\frac{\Delta x}{\Delta t} \approx v_1, \quad \frac{\Delta v}{\Delta t} = g$$

In the limit of continuous observations, i.e. for $\Delta \to 0$, the finite difference equation turns into a differential equation

$$\frac{dx}{dt} \approx v_1, \quad \frac{dv}{dt} = g$$

Historically, differential equations and their solutions have a strong link to dynamical systems. Then the independent variable is time and the derivative is w.r.t. time. Of course, any other independent variable can be chosen.

Definition 1 (Differential equations)**.** *A* differential equation *is an equation which contains (one or several) derivatives (d/dx_i) of an unknown function $f(\vec{x})$ w.r.t. the independent variables $\vec{x} = (x_1, x_2 \dots x_n)$.*
If, furthermore,

- *all derivatives are w.r.t. a single variable only the differential equation is called an* ordinary *differential equation (ODE), e.g.* $f'(x) = -3f(x)$.
- *derivatives w.r.t. different observables occur it is called a* partial *differential equation, e.g. heat conduction in a one-dimensional bar (with* λ *heat conductivity,* c *specific heat and* ρ *density of a material).*

$$\frac{\partial T(x,t)}{\partial t} = \frac{\lambda}{c\,\rho}\frac{\partial^2 T(x,t)}{\partial t^2}$$

If a differential equation holds for a function $f(\vec{x})$ *this function is called a* solution *of the differential equation.*

Notation

- The (total) derivative of a function $f(x)$ w.r.t. the independent variable x is best denoted by $\frac{df}{dx}$.
- The partial derivative of a function $f(\vec{x})$ w.r.t. an independent variable x_i is denoted by $\frac{\partial f}{\partial x}$.
- Historically, the derivative w.r.t. the independent variable t (time) is sometimes denoted by a dot, e.g. $v = \dot{x}(t)$.
- Similarly, a derivative w.r.t. the independent variable x is denoted by an apostrophe, e.g. $f'(x)$.

1.2 Classification of differential equations

Definition 2. *Differential equations can be classified w.r.t. the following definitions:*

(1) **Order:** *A differential equation is said to be of* order n *if the highest derivative involved is the* $n-th$ *derivative of the function w.r.t. the independent variable*

(2) **Linear:** *A differential equation is called* linear *if the unknown function* $f(x)$ *occurs to first order only, e.g.* $df(x)/dx = f(x)$.
Nonlinear: *It is called* nonlinear*, if the unknown function occurs to higher order, e.g.* $df(x)/dx = f^3(x)$*, or inside of a nonlinear function, e.g.* $df(x)/dx = \log(f(x))$.

(3) **Explicit:** *A differential equation of order* n *is called* explicit *if the highest order derivative can be extracted such that*

$$\frac{d^n f(x)}{dx^n} = F\left(f(x), \frac{df}{dx}, \frac{d^2 f}{dx^2}, \ldots, \frac{d^{n-1} f}{dx^{n-1}}; x\right)$$

Implicit: *It is called* implicit *if the differential equation is given by a functional expression of the form*

$$F\left(f(x), \frac{df}{dx}, \frac{d^2 f}{dx^2}, \ldots, \frac{d^n f}{dx^n}; x\right) = 0$$

(4) **Autonomous:** *An explicit differential equation is called* autonomous *if the function $F(\ldots)$ does not depend on the independent variable (x) directly, i.e. $F(\ldots) \neq F(\ldots, x)$. E.g. $df/dx = -3f(x)$*

(5) **Homogeneous:** *A differential equation is called* homogeneous *if all its terms depend on the dependent variable or one of its derivatives. Equivalently, one can say that the differential equation does not contain a perturbation (s.b.). For instance, the differential equation $f'(x) = \alpha(x)f(x)$ is homogeneous but not autonomous.*
Inhomogeneous: *Otherwise, it is called inhomogeneous, e.g. $df/dx = f(x) + g(x)$. $g(x)$ is called a perturbation.*

(6) **Separable:** *A differential equation is called* separable *if it can be written as a product of functional expressions of the independent variable (x) and of the dependent variable $f(x)$*

$$\frac{df(x)}{dx} = g(x) \cdot h(f(x))$$

Example 1. *Consider the following differential equations:*

Definition 3 (Initial value problem (IVP))**.** *An initial value problem is defined by a differential equation of order n and a set of n so-called* initial conditions $f^{(i)}(x_0) = y_0^i$.

Definition 4. *A boundary value problem is defined by a differential equation of order n and up to $i \in [1, n]$ boundary values $(x_i, f(x_i)) = (x_i, y_i)$. This defines n algebraic equations $f(x_i) = y_i$ which can be used to fix the n general integration constants.*

1.3 Solving differential equations

There are different approaches (algorithms, receipies) for solving different classes of ordinary differential equations. Solving a differential equation implies some form of integration. To obtain a unique solution additional information is required (initial values or boundary conditions). They are needed to fix general integration constants.

(A) General methods for solving ODEs

1.3.1 Educated guess

The most successful (but unsystematic) approach is to guess a solution and prove that it is correct. This can be done easily as forming derivatives is a simple operation.

Example 2. *Consider the differential equation $y''(x) + y(x) - x = 0$. The solution is simple: $y(x) = x$. Test it! An initial value problem is set if two constraints are given, e.g. $y(0) = 0$ and $y'(0) = 1$.*

It is quite common to guess a solution and prove that it is correct.

1.3.2 Substitution

Consider a differential equation $\frac{dy}{dx} = f(h(x, y))$. Sometimes integrals and differential equations can be solved more easily if a term including the dependent variable $(y(x))$ is substituted by a different variable $u = h(x, y)$. A new differential equation is generated by

$$\frac{du}{dx} = \frac{\partial h(x,y)}{\partial x} + \frac{\partial h(x,y)}{\partial y}\frac{dy}{dx} = \frac{\partial h(x,y)}{\partial x} + \frac{\partial h(x,y)}{\partial y} f(u)$$

Then the new differential equation is solved for $u(x)$ and from $u = h(x, y)$ the solution $y(x)$ can be found.

Example 3. *Consider the differential equation $\frac{dy(x)}{dx} = f(ax + by(x) + c)$. Then $u = h(x, y(x)) = ax + by(x) + c$. The new differential equation reads*

$$\frac{du}{dx} = a + b\frac{dy(x)}{dx} = a + bf(u)$$

Solve the new ODE for $u(x)$ and substitute the solution into $u = h(x, y)$ to solve for $y(x)$.

(B) Solving first order ODEs

A general first order ODE can be written as

$$\frac{df(x)}{dx} = F(x, f(x)) \qquad \text{or} \qquad A(x, f(x))dx + B(x, f(x))df(x) = 0$$

1.3.3 Solving separable homogeneous first order ODEs

Separable homogeneous ODEs can be systematically solved by separating the dependent and the independent variable. Both linear and non-linear

ODEs can be treated in this way.

$$\boxed{\frac{df(x)}{dx} = g(x) \cdot h(f(x))}$$

Use the following solution method to determine a solution:

(1) Separation of variables: $\frac{df(x)}{h(f(x))} = g(x)dx$.

(2) Integration of both sides:

$$\int_{f(x_0)}^{f(x)} \frac{1}{h(\tilde{f}(x))} d\tilde{f}(x) = \int_{x_0}^{x} g(\tilde{x})d\tilde{x}$$

(3) Solve for f(x): $f(x) = \dots$

(4) Determine coefficients by using initial condition $f(x_0) = y_0$.

Simple case: $h(x) = id(x) = x$. Then the general solution is an exponential:

Example 4. *Discuss the following examples:*

1. *Let $\dot{P}(t) = \lambda P(t)$ with $P(t) > 0$ describe the growth of a population of bacteria with a growth rate λ. Initially, the population was $P(0) = P_0$.*

   (1) Separation of variables: $dP(t)/P(t) = \lambda dt$.

   (2) Integration of both sides:

   $$\int \frac{dP(t)}{P(t)} = \int \lambda dt \;\Leftrightarrow \ln(P(t)) - \ln(P(t_0)) = \lambda(t - t_0)$$

   (3) Solve for P(t): $P(t) = P_(t_0)e^{\lambda(t-t_0)}$

   (4) Determine coefficients by using the initial condition.

   This example can be solved for any initial condition (t_0, P_0).

2. *Now let $y'(x) = xy(x)$ with $y(x) > 0$. Use the algorithm:*

   (1) Separation of variables: $dy(x)/y(x) = xdx$.

   (2) Integration of both sides:

   $$\int \frac{dy(x)}{y(x)} = \int xdx \;\Leftrightarrow \ln(y(x)) - \ln(y(x_0)) = 1/2(x^2 - x_0^2)$$

   (3) Solve for y(x): $y(x) = y(x_0)e^{(x^2-x_0^2)/2}$

   (4) Determine coefficients by using the initial condition.

   This example can be solved for any initial condition (x_0, y_0).

3. *Let* $y'(x) = \sqrt{y(x)}$ *with the initial condition* $y(0) = 0$.
   Trivial solution: $y(x) = 0 \quad \forall x \in \mathbb{R}_0^+$.
   Nontrivial solution:

$$y_\lambda(x) = \begin{cases} 0 & \text{for } \; 0 \leq x \leq \lambda \\ \frac{1}{4}(x-\lambda)^2 & \text{for } \; x > \lambda \end{cases}$$

   There is an infinite number of solutions for this initial value ($\lambda \in \mathbb{R}_0^+$)

4. *Capacitor discharge over a resistor (see simulation, Problem set 2)*

1.3.4 Solving *exact* ordinary differential equations

Definition 5. *The total differential* dU *of a function* $U(x,y)$ *of two independent variables* x *and* y *is defined by*

$$dU = \frac{\partial U(x,y)}{\partial x}dx + \frac{\partial U(x,y)}{\partial y}dy$$

Definition 6. *A first degree first order ODE is called* exact *if with* $y = f(x)$

$$A(x,y)dx + B(x,y)dy = 0 \qquad \text{and} \qquad \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \quad (*)$$

If an ODE is exact it represents the total differential $dU(x,y)$ of a potential function $U(x,y)$. Thus,

$$A(x,y)dx + B(x,y)dy = dU(x,y) = 0 \qquad \text{with}$$

$$A(x,y) = \frac{\partial U(x,y)}{\partial x} \quad \text{and} \quad B(x,y) = \frac{\partial U(x,y)}{\partial y}$$

Note that then the condition (*) is equivalent to the interchange of partial derivations

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \quad \Leftrightarrow \quad \frac{\partial^2 U(x,y)}{\partial y \partial x} = \frac{\partial^2 U(x,y)}{\partial x \partial y}$$

We can solve the ODE with the following solution method:

(1) Check thet the ODE is exact, i.e. that $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$

(2) Construct the potential function $U(x.y)$:

   (2.1) Integrating $dU(x,y) = 0$ results in $U(x,y) = C_1$

   (2.2) Integrating $A(x,y) = \frac{\partial U(x,y)}{\partial x}$ w.r.t. x results in

$$U(x,y) = \int A(x,y)dx + F(y)$$

(2.3) Plug the intermediate result for $U(x,y)$ into $B(x,y) = \frac{\partial U(x,y)}{\partial y}$, i.e. derive w.r.t. y, and

(2.4) Determine $F(y)$ by solving a differential equation for $F(y)$

(3) Solve for $y = f(x)$.

Example 5. *Solve* $2x\frac{dy}{dx} + 5x + 2y = 0$. *Why does separation of variables not work here?*

(0) Rewrite the ODE as $(5x+2y)\ dx\ +\ 2x\ dy = 0$ *with*
$A(x,y) = 5x + 2y$ *and* $B(x,y) = 2x$.
Variables cannot be separated here because $A(x,y) \neq A(x)$.

(1) Check thet the ODE is exact: $\frac{\partial A}{\partial y} = 2$ *and* $\frac{\partial B}{\partial x} = 2$ *(ok)*

(2) Construct the potential function $U(x.y)$*:*

(2.1) Integrating $dU(x,y) = 0$ *results in* $U(x,y) = C_1$

(2.2) Integrating $A(x,y) = \frac{\partial U(x,y)}{\partial x}$ *w.r.t.* x *results in*

$$C_1 = U(x,y) = \int (5x + 2y)dx + F(y) = \frac{5}{2}x^2 + 2xy + F(y)$$

(2.3) Plug the intermediate result for $U(x,y)$ *into* $B(x,y) = \frac{\partial U(x,y)}{\partial y}$, *i.e. derive w.r.t.* y,

$$\frac{\partial U(x,y)}{\partial y} = 2x + \frac{dF(y)}{dy} = B(x,y) = 2x$$

(2.4) Determine $F(y)$ *by solving the differential equation* $\frac{dF(y)}{dy} = 0$, *i.e.* $F(y) = C_2$.

(3) From (2.2)

$$C_1 = \frac{5}{2}x^2 + 2xy + C_2 \quad \Rightarrow \quad y(x) = \frac{C_1 - C_2}{2x} - \frac{5}{4}x = \frac{C}{2x} - \frac{5}{4}x$$

(4) Test if the function works in the ODE!

$$2x\left(-\frac{C}{2x^2} - \frac{5}{4}\right) + 5x + 2\left(\frac{C}{2x} - \frac{5}{4}x\right) = 0$$

These kind of integrals often occur in thermodynamics (heat engines, etc.).

1.3.5 Solving special *inexact* ordinary differential equations

Definition 7. *A first degree first order ODE is called* inexact *if with* $y = f(x)$

$$A(x,y)dx + B(x,y)dy = 0 \qquad \text{but} \qquad \frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x} \quad (*)$$

Such an ODE can be always made exact by multiplying an *integrating factor* $\mu(x,y)$ such that

$$\frac{\partial(\mu(x,y)A(x,y))}{\partial y} = \frac{\partial(\mu(x,y)B(x,y))}{\partial x}$$

This equation is only practical if μ depends on x or y, say $\mu = \mu(x)$. Then one can calculate the integrating factor from

$$\mu(x)\frac{\partial A(x,y)}{\partial y} = \mu(x)\frac{\partial B(x,y)}{\partial x} + B(x,y)\frac{\partial \mu(x)}{\partial x}$$

This can be solved by separation of variables

$$\frac{d\mu}{\mu} = \frac{1}{B(x,y)}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) dx =: g(x)dx$$

and results in an exponential solution

$$\mu(x) = e^{\int g(x)dx} = e^{\int \frac{1}{B(x,y)}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)dx}$$

Example 6. *Consider the ODE* $(4x + 3y^2)\ dx\ +\ 2xy\ dy = 0$.
This implies $A(x,y) = 4x + 3y^2$ *and* $B(x,y) = 2xy$. *With this*

$$\frac{\partial A(x,y)}{\partial y} = 6y \qquad , \qquad \frac{\partial B(x,y)}{\partial x} = 2y$$

Hence this ODE is not exact. However,

$$\frac{1}{B(x,y)}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) = \frac{2}{x}$$

only depends on x. *Thus an integrating factor can be found such that*

$$\mu(x) = e^{2\int \frac{dx}{x}} = e^{2\ln(x)} = x^2$$

Now the ODE is solved like an exact ODE.

1.3.6 Solving linear inhomogeneous first order ODEs

Definition 8. *A linear inhomogeneous first order ODE is of the form*

$$\frac{dy(x)}{dx} + p(x)y = q(x) \qquad or \qquad dy + (p(x)y - q(x))dx = 0$$

The term $q(x)$ which makes the ODE inhomogeneous is called a perturbation*.*

It is a special case of an inexact ODE as the integrating factor is always a function of x alone. An integrating factor $\mu(x)$ can be chosen such that

$$\mu(x)\frac{dy(x)}{dx} + \mu(x)p(x)y = \frac{d}{dy}[\mu(x)y] = \mu(x)q(x)$$

This equation contains two pieces of information:

(i) Solve for $\mu(x)$ from

$$\frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu\frac{dy}{dx} + \mu p(x)y$$

$$\Rightarrow \quad \frac{d\mu}{dx} = \mu p(x) \quad \Rightarrow \quad \mu(x) = e^{\int p(x)dx}$$

(ii) Integrate such that $\mu(x)y = \int \mu(x)q(x)dx$

This method is also known as the variation of constants

Example 7. *Solve the ODE*

$$\frac{dy}{dx} + 2xy = 4x$$

check: the ODE is linear, first order and inhomogeneous.

(i) The integrating factor is given by

$$\mu(x) = e^{\int 2x dx} = e^{x^2}$$

(ii) Integrating

$$ye^{x^2} = 4\int xe^{x^2}dx = 2e^{x^2} + C \qquad \Rightarrow \qquad y(x) = 2 + Ce^{-x^2}$$

1.3.7 General solution for inhomogeneous first order ODEs

Linear inhomogeneous ODEs can be systematically solved if the corresponding homogeneous solution can be solved.

We consider linear ODEs of the following form:

$$\boxed{\frac{dy(x)}{dx} + p(x)y = q(x)}$$

$q(x)$ is called a *perturbation* or inhomogeneity.

Theorem 1. *The solution $y(x) = y_h(x) + y_p(x)$ of a linear inhomogeneous ODE is given by the superposition of the general solution of the corresponding homogeneous ODE $y_h(x)$ and a particular solution $y_p(x)$ of the inhomogeneous ODE.*

Method for solution:

(1) Solve the corresponding homogeneous ODE

(2) Find a particular solution of the inhomogeneous ODE, e.g. by calculating the integrating factor.

Example 8. *Consider the following IVP based on a linear inhomogeneous ODE:*

$$y' = \frac{y}{x} + 5x \qquad \text{with} \quad x > 0,\ y(1) = 0$$

Comparing with our general form we read off:

$$p(x) = -\frac{1}{x} \qquad \text{and} \quad q(x) = 5x$$

Since $q(x) \neq 0$ the ODE is inhomogeneous.

Method for solution.

(0) The general solution $y(x) = y_h(x) + y_p(x)$ is given as the superposition of the general solution of the corresponding homogeneous ODE $y_h(x)$ and a particular solution $y_p(x)$ of the inhomogeneous ODE.

(1) Solution of the homogeneous ODE $y' = \frac{y}{x}$ by separation:

$$\frac{dy}{y} = \frac{dx}{x} \quad \Rightarrow \quad \ln(y) - \ln(y_0) = \ln(x)\,|e^{\cdots} \quad \Rightarrow \quad y_h = Cx \quad \text{with } C \in \mathbb{R}$$

(2) Finding a particular solution of the inhomogeneous equation:

(2.1) Determine the integrating factor

$$\mu(x) = e^{\int p(x)dx} = e^{-\int 1/x dx} = e^{-\ln(x)dx} = \frac{1}{x}$$

(2.2) Integrate

$$y_p(x) = \frac{1}{\mu(x)} \int \mu(x)q(x)dx = x \int \frac{1}{x} 5x dx = 5x^2$$

(3) General solution of the inhomogeneous equation: $y(x) = y_h + y_p = Cx + 5x^2$.

(4) Determine the constant to solve the initial value problem: $y(1) = 0 \Rightarrow C = -5$. *So the final solution is*

$$y(x) = 5x^2 - 5x$$

Example 9. *Consider the linear inhomogeneous ODE:*

$$y' + 5y = \sin(x)$$

Comparing with our general form we read off:

$$p(x) = 5 \qquad \text{and} \quad q(x) = \sin(x)$$

Since $q(x) \neq 0$ *the ODE is inhomogeneous. As* $p(x)$ *is constant it is called an ODE with constant coefficients. This is a particular simple case because the homogeneous solution and the integrating factor are simple exponential functions.*

Method for solution.

(0) The general solution $y(x) = y_h(x) + y_p(x)$ *is given as the superposition of the general solution of the corresponding homogeneous ODE* $y_h(x)$ *and a particular solution* $y_p(x)$ *of the inhomogeneous ODE.*

(1) Solution of the homogeneous ODE $y' = -5yx$ *by separation:*

$$\frac{dy}{y} = \frac{dx}{x} \quad \Rightarrow \quad \ln(y) - \ln(y_0) = -5x \,|e^{\cdots} \quad \Rightarrow \quad y_h = Ce^{-5x} \quad \text{with } C \in \mathbb{R}$$

(2) Finding a particular solution of the inhomogeneous equation:

(2.1) Determine the integrating factor

$$\mu(x) = e^{\int p(x)dx} = e^{\int 5dx} = e^{5x}$$

(2.2) Integrate

$$y_p(x) = \frac{1}{\mu(x)} \int \mu(x)q(x)dx = e^{-5x} \int e^{5x} \sin(x)dx = \frac{5}{26} \sin(x) - \frac{1}{26} \cos(x)$$

(3) General solution of the inhomogeneous equation: $y(x) = y_h + y_p = Ce^{-5x} + \frac{5}{26} \sin(x) - \frac{1}{26} \cos(x)$

Integration is done by performing an integration by parts twice. Alternatively, a general ansatz $y_p = A\sin(x) + B\cos(x)$ *can be used.*

1.3.8 Existence of solutions

Different situations can occur:

1. The ODE cannot be solved. No solution exists.
2. The ODE can be solved, i.e. it exists a formal solution, but the initial value problem (IVP) or boundary value problem (BVP) cannot be solved.
3. The IVP can be solved, but not uniquely. Then there exists a solution space of dimension $d > 0$.
4. The IVP can be solved uniquely.

(B) Solving higher order ODEs

1.4 Solving linear second order ODEs with constant coefficients

1.4.1 General method

We consider ODEs of the following form:

$$y'' + ay' + by = g(x)$$

Solution method:

(1) Solution of the homogeneous ODE $y'' + ay' + by = 0$.

(1.1) Use the exponential Ansatz $y = e^{\lambda x}$. By inserting it into the ODE one obtains the characteristic polynomial

$$P(\lambda) = \lambda^2 + a\lambda + b = 0$$

(1.2) Calculate the roots of the polynomial (either real or complex numbers)

$$\lambda_{1/2} = -\frac{1}{2}\left(a \pm \sqrt{a^2 - 4b}\right)$$

Every root corresponds to a solution of the homogeneous ODE. These solutions $(y_1(x), y_2(x))$ are called *fundamental* solutions or *basis* solutions of the homogeneous ODE if they are linearly independent, i.e. if the *Wronski determinant* (also called the *Wronskian*) is nonzero

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} \neq 0$$

(1.3) Construct the corresponding fundamental solutions for all roots of the characteristic polynomial. This depends on the discriminant $D = a^2 - 4b$

I $D > 0$: The characteristic polynomial has two distinct real roots: $\lambda_1 \neq \lambda_2$. Then the two solutions are exponentials:

$$y_1(x) = e^{\lambda_1 x} \qquad \text{and} \qquad y_2(x) = e^{\lambda_2 x}$$

II $D = 0$: The characteristic polynomial has a degenerate real solution $\lambda_1 = \lambda_2 = \lambda$. Then the two solutions need to be constructed as linear independent functions

$$y_1(x) = e^{\lambda x} \qquad \text{and} \qquad y_2(x) = xe^{\lambda x}$$

III $D < 0$: The characteristic polynomial has two distinct complex roots which are complex conjugates to each other: $\lambda_1 \neq \lambda_2$ with $\lambda_{1/2} = \alpha \pm \mathbf{j}\beta$. While the real part brings about a damping / growth factor, the imaginary part results in an oscillation:

$$y_1(x) = e^{\alpha x}\cos(\beta x) \qquad \text{and} \qquad y_2(x) = e^{\alpha x}\sin(\beta x)$$

(1.4) The general solution of the homogeneous ODE is given by the sum of both solutions (with $C_1, C_2 \in \mathbb{R}$)

$$y_h = C_1(x)y_1(x) + C_2(x)y)2(x)$$

(2) Find a particular solution of the inhomogeneos equation

(2.V1) Full formal approach (once in a lifetime):

(2.V1.1) Set up a system of two coupled equations for the derivatives of two x-dependent constants for the particular solution of the inhomogeneous equation. This can be done by

(2.V1.1.1) Assume that the constants are x-dependent

$$y_h = C_1(x)y_1(x) + C_2(x)y_2(x)$$

(2.V1.1.2) Calculate the derivatives:

$$\begin{aligned}
y_h(x) &= C_1(x)y_1(x) + C_2(x)y_2(x) \\
y_h'(x) &= C_1'(x)y_1(x) + C_1(x)y_1'(x) + \\
&\quad C_2'(x)y_2(x) + C_2(x)y_2'(x) \\
y_h''(x) &= C_1''(x)y_1(x) + C_1'(x)y_1'(x) + \\
&\quad C_1'(x)y_1'(x) + C_1(x)y_1''(x) + \\
&\quad C_2''(x)y_2(x) + C_2'(x)y_2'(x) + \\
&\quad C_2'(x)y_2'(x) + C_2(x)y_2''(x)
\end{aligned}$$

(2.V1.1.3) Make a simplifying choice:

$$\begin{aligned} 0 &= C_1'(x)y_1(x) + C_2'(x)y_2(x)\ \forall x \qquad (*) \quad \Rightarrow \\ 0 &= C_1''(x)y_1(x) + C_1'(x)y_1'(x) + \\ & \quad C_2''(x)y_2(x) + C_2'(x)y_2'(x) \end{aligned}$$

This results in

$$\begin{aligned} y_h(x) &= C_1(x)y_1(x) + C_2(x)y_2(x) \\ y_h'(x) &= C_1(x)y_1'(x) + C_2(x)y_2'(x) \\ y_h''(x) &= C_1'(x)y_1'(x) + C_1(x)y_1''(x) + \\ & \quad C_2'(x)y_2'(x) + C_2(x)y_2''(x) \end{aligned}$$

(2.V1.1.4) Insert this ansatz into the homogeneous ODE

$$y'' + ay' + by = g(x)$$

and sort the terms:

$$\begin{aligned} C_1 \quad & [y_1''(x) + ay_1'(x) + by_1(x)] = 0 \\ C_2 \quad & [y_2''(x) + ay_2'(x) + by_2(x)] = 0 \end{aligned}$$

So only a simple equation remains:

$$C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = g(x) \quad (**)$$

(2.V1.1.5) Arrange the two equations (*) and (**) in a system of coupled equations:

$$\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix} \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}$$

Note that the matrix is the Wronski matrix.

(2.V1.2) Solve the system of equations (e.g. using Cramer's rule. Then the Wronzkian is used)

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix}$$

and calculate the two x-dependent constants by integration:

$$\begin{aligned} C_1(x) &= \int \frac{1}{W(x)} \left(-y_2(x)g(x)\right) dx \qquad \text{and} \\ C_2(x) &= \int \frac{1}{W(x)} \left(y_1(x)g(x)\right) \end{aligned}$$

(2.V1.3) Construct the particular solution from the x-dependent constants and the solutions of the homogeneous system as

$$y_p = C_1(x)y_1(x) + C_2(x)y_2(x)$$

(2.V1) Short version:

(2.V1.1a) Calculate the Wronzkian:

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix}$$

(2.V1.2a) Use the semi-finished formulae

$$\begin{aligned} C_1(x) &= \int \frac{1}{W(x)} \left(-y_2(x)g(x)\right) dx \qquad \text{and} \\ C_2(x) &= \int \frac{1}{W(x)} \left(y_1(x)g(x)\right) \end{aligned}$$

(2.V1.3a) Construct the particular solution from the x-dependent constants and the solutions of the homogeneous system as

$$y_p = C_1(x)y_1(x) + C_2(x)y_2(x)$$

(2.V2) **Alternative approach:**
Select an Ansatz which corresponds to the structure of the inhomogeneous term and fix the coefficients:

$g(x)$	Conditions	Ansatz for $y_p(x)$
ae^{rx}	$r \neq \lambda_i \ \forall i$	be^{rx}
	$r = \lambda_i$ (single root)	bxe^{rx}
	$r = \lambda_i$ (double root)	bx^2e^{rx}
$a_1 \cos(rx)$ $+a_2 \sin(rx)$	$r \neq \lambda_i \ \forall i$	$b_1 \cos(rx) + b_2 \sin(rx)$
	$r = \lambda_i$ (single root)	$x\ [b_1 \cos(rx) + b_2 \sin(rx)]$
	$r = \lambda_i$ (double root)	$x^2\ [b_1 \cos(rx) + b_2 \sin(rx)]$
$\sum_{i=0}^N a_i x^i$	$b \neq 0$	$\sum_{i=0}^N b_i x^i$
	$a \neq 0, b = 0$	$x \sum_{i=0}^N b_i x^i$
	$a = b = 0$	$x^2 \sum_{i=0}^N b_i x^i$

Example 10. *Consider the ODE* $y'' + 2y' - 8y = x$

(1) Solution of the homogeneous ODE $y'' + 2y' - 8y = 0$.

(1.1) Use the exponential Ansatz $y = e^{\lambda x}$. *By inserting it into the ODE one obtains the characteristic polynomial*

$$P(\lambda) = \lambda^2 + 2\lambda - 8 = 0$$

(1.2) Calculate the roots of the polynomial (either real or complex numbers)

$$\lambda_1 = 2 \qquad \lambda_2 = -4$$

Every root represents one solution of the homogeneous ODE.

(1.3) Construct the corresponding solutions depending on the discriminant $D = a^2 - 4b$

I $D > 0$: The characteristic polynomial has two distinct real roots: $\lambda_1 \neq \lambda_2$. Then the two solutions are exponentials:

$$y_1(x) = e^{2x} \quad y_2(x) = e^{-4x}$$

(1.4) The general solution of the homogeneous ODE is given by the sum of both solutions (with $C_1, C_2 \in \mathbb{R}$)

$$y_h = C_1 e^{2x} + C_2 e^{-4x}$$

(2) Find a particular solution of the inhomogeneos equation

(2.V1) Formal approach:

(2.V1.1) Instead of an integrating factor calculate the determinant of the Wronski matrix, the so-called Wronskian:

$$W(x) = \begin{vmatrix} e^{2x} & e^{-4x} \\ 2e^{2x} & -4e^{-4x} \end{vmatrix} = -6e^{-2x}$$

(2.V1.2) Calculate two x-dependent constants for the particular solution of the inhomogeneous equation using the solutions of the homogeneous system:

$$\begin{aligned} C_1(x) &= \int \frac{1}{-6} e^{2x} \left(-e^{-4x} x\right) dx = -\frac{1}{12} e^{-2x} \left(x + \frac{1}{2}\right) \\ C_2(x) &= \int \frac{1}{-6} e^{2x} \left(e^{2x} x\right) = \frac{1}{24} e^{4x} \left(-x + \frac{1}{4}\right) \end{aligned}$$

(2.V1.3) Construct the particular solution from the x-dependent constants and the solutions of the homogeneous system as

$$y_p = C_1(x) y_1(x) + C_2(x) y_2(x) = -\frac{1}{8} x - \frac{1}{32}$$

(2.V2) Alternative approach: Select an Ansatz which corresponds to the structure of the inhomogeneous term and fix the coefficients: The relevant entry of the table is

$g(x)$	*Ansatz for* $y_p(x)$
$\sum_{i=0}^{N} a_i x^i$	$\sum_{i=0}^{N} b_i x^i$

with $N = 1$

$$\begin{aligned} y(x) &= b_1 x + b_0 \\ y'(x) &= b_1 \\ y''(x) &= 0 \end{aligned}$$

Inserting this ansatz into the ODE gives

$$2b_1 - 8b_1 x - 8b_0 = x \qquad \Rightarrow \qquad (2b_1 - 8b_0) - (8b_1 + 1)x = 0$$

This equation must hold for all $x \in \mathbb{R}$. Hence each bracket must vanish independently and $b_1 = -1/8$, $b_0 = 1/4b_1 = -1/32$.

1.4.2 The harmonic oscillator

We study the linear second order ODE which describes small oscillations around an equilibrium value. For instance, mechanical motion is goverened by the Newton law (in one dimension)

$$ma = m\frac{d^2x(t)}{dt} = \sum_i F_i$$

(with m: mass, a: acceleration, x: position, t: time, F: force)
This is a second order ODE in $x(t)$.
For different sums of forces different solutions arise. We only consider forces which are maximally linear in $x(t)$ such that a linear ODE holds.
The defining force of the harmonic oscillator follows from Hooke's law

$$F = -Dx = -m\omega^2$$

Other forces may be added.

Name of oscillator	additional F_i	ODE (divided by m)
Free undamped	0	$\frac{d^2x(t)}{dt^2} + \omega_0^2 x = 0$
Free damped	$-bv(t)$	$\frac{d^2x(t)}{dt^2} + 2\delta\frac{dx(t)}{dt} + \omega_0^2 x = 0$
Harmonically driven and damped	$F_0 \sin(\Omega t)$ $-bv(t)$	$\frac{d^2x(t)}{dt^2} + 2\delta\frac{dx(t)}{dt} + \omega_0^2 x = F_0 \sin(\Omega t)$

δ is called the damping factor and intentionally defined with a factor 2 as $2\delta = b/m$. $\omega_0^2 = D/m$ is called the eigenfrequency of the oscillation.

Free damped oscillator: A good example case for solving a homogeneous linear second order ODE:

$$\frac{d^2x(t)}{dt^2} + 2\delta\frac{dx(t)}{dt} + \omega_0^2 x = 0$$

We first derive the characteristic polynomial with the ansatz $x(t) = e^{\lambda t}$:

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0 \qquad \Rightarrow \qquad \lambda_{1/2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

The solutions depend on the relative strength of the damping δ w.r.t. the eigenfrequency ω_0. Three cases exist:

(1) $\delta < \omega_0$: Underdamped regime with damped but oscillating solution (dt. Schwingfall)

(2) $\delta = \omega_0$: Critically damped regime (dt. Aperiodischer Grenzfall)

(3) $\delta < \omega_0$: Overdamped regime without oscillations (dt Kriechfall).

We calculate the solution for those three cases.

(1) The Ansatz for the underdamped regime is

$$x(t) = e^{-\delta t}[C_1 \sin(\omega_d t) + C_2 \cos(\omega_d t)]$$

with $\omega_d = \sqrt{\omega_0^2 - \delta^2}$.

(2) This is a degenerate case with double roots: The Ansatz for the critically damped regime is now given by

$$x(t) = (C_1 t + C_2)e^{-\delta t}$$

(3) The Ansatz for the overdamped regime is a linear superposition of damped exponentials

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Two initial conditions are needed to fix the two constants.

Harmonically driven damped oscillator: The harmonically driven oscillator is described by

$$\frac{d^2x(t)}{dt^2} + 2\delta\frac{dx(t)}{dt} + \omega_0^2 x = \frac{F_0}{m}\sin(\Omega t)$$

We have already found the general solution of the homogeneous equation.

The inhomogeneous equation can be solved with a real ansatz or, simpler, with a complex ansatz function. Note that

$$e^{j\Omega t} = \cos(\Omega t) + j\sin(\Omega t)$$

Since the ODE is linear we can also solve a complex version

$$\frac{d^2x(t)}{dt^2} + 2\delta\frac{dx(t)}{dt} + \omega_0^2 x = \frac{F_0}{m}e^{j\Omega t}$$

and take the imaginary part of the solution. We use the ansatz

$$\begin{aligned} x_p(t) &= Ae^{j(\Omega t+\phi)} \\ \frac{dx_p(t)}{dt} &= Aj\Omega e^{j(\Omega t+\phi)} \\ \frac{d^2x_p(t)}{dt^2} &= -A\Omega^2 e^{j(\Omega t+\phi)} \end{aligned}$$

Inserting this into the complex ODE results in

$$-\Omega^2 e^{-j\phi} + 2\delta\Omega j e^{-j\phi} + \omega_0^2 e^{-j\phi} = \frac{F_0}{mA}$$

$$(\Omega^2 - \omega^2) + j(2\delta\Omega) = F_0/Ae^{j\phi}$$

We read this as a complex number:

$$\left(\frac{F_0}{mA}\right)^2 = (\Omega^2 - \omega^2)^2 + (2\delta\Omega)^2 \qquad \Rightarrow$$

$$\begin{aligned} A(\Omega) &= \frac{F_0}{m\sqrt{(\Omega^2-\omega^2)^2 + 4\delta^2\Omega^2}} \\ \tan(\phi) &= \frac{2\delta\Omega}{\Omega^2-\omega^2} \\ \phi(\Omega) &= \begin{cases} \arctan\left(\frac{2\delta\Omega}{\Omega^2-\omega^2}\right) & (\Omega < \omega_0) \\ \frac{\pi}{2} & (\Omega = \omega_0) \\ \arctan\left(\frac{2\delta\Omega}{\Omega^2-\omega^2}\right) + \pi & (\Omega > \omega_0) \end{cases} \end{aligned}$$

Resonance Discussion of $A(\Omega)$ shows resonant behavior. We find the resonance frequency by searching for the minimum of the denominator. This results in $\Omega_r = \sqrt{\omega_0^2 - 2\delta^2}$.

Note the relation $\Omega_r < \omega_d < \omega_0$, i.e. for a damped oscillator the resonance frequency is smaller as the damped oscillation frequency and of the eigenfrequency of the oscillator.

1.4.3 Generalization for linear ODEs of n-th order with constant coefficients

We consider ODEs of the following form (with $a_i \in \mathbb{R}$):

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x)$$

Solution method:

(1) Find a solution of the homogeneous ODE $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$.

(1.1) Use the exponential Ansatz $y = e^{\lambda x}$. By inserting it into the ODE one obtains the characteristic polynomial

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$$

(1.2) Calculate the roots of the polynomial (either real or complex numbers) Every root represents one solution of the homogeneous ODE.

(1.3) Construct the corresponding solutions depending on the character of the root:

I Single real or complex root: The corresponding solution is a real or complex exponential:

$$y(x) = e^{\lambda x}$$

II Multiplicity of roots: The characteristic polynomial has degenerate solutions with a multiplicity k: Then the k solutions need to be constructed as linear independent functions

$$y_j^{[k]}(x) = x^j e^{\lambda x} \quad j \in [0, k-1]$$

(1.4) The general solution of the homogeneous ODE is given by the sum of all solutions (with $C_i \in \mathbb{R}$)

$$y_h = \sum_{i=1}^{n} C_i y_i(x)$$

(2) Find a particular solution of the inhomogeneos equation

(2.V1) Formal approach:

(2.V1.1) Solve the following system of equations for the derivatives of the x-dependent constants $C'(x)$

$$\begin{pmatrix} y_1 & y_2 & \ldots & y_n(x) \\ y_1' & y_2' & \ldots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} C_1'(x) \\ C_2'(x) \\ \vdots \\ C_n'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(x) \end{pmatrix}$$

(2.V1.2) Calculate all x-dependent constants for the particular solution of the inhomogeneous equation by integrating the expressions for $C'(x)$.

(2.V1.3) Construct the particular solution from the x-dependent constants and the solutions of the homogeneous system as

$$y_p(x) = \sum_{i=1}^{n} C_i(x) y_i(x)$$

(2.V2) Alternative approach: Select an Ansatz which corresponds to the structure of the inhomogeneous term and compare the coefficients.

1.4.4 Alternative treatment of linear ODEs of n-th order

Expansion of a higher order explicit ODE. An ODE of order $n > 1$ with dependent variable $y(x)$ and independent variable x

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x)$$

can be expanded as a set of n ODEs of order one by substitution: $(n-1)$ new dependent variables $z_i(x)$ are defined by the differential equations $z_i = d^i y/dx^i$. Then the original ODE reads

$$\frac{dz_{n-1}}{dx} + a_{n-1}z_{(n-1)} + \ldots + a_1 z_1 + a_0 y(x) = g(x)$$

Together with the $(n-1)$ defining ODEs for $z_i(x)$ a aystem of n coupled linear first order ODEs has been obtained. This can be solved e.g. numerically.

1.5 Systems of coupled ODEs