b) $y(2t^2y^2+1)\dfrac{dy}{dt} + t(y^4+1) = 0$

(0) Re-write equation in standard form:

$$\underbrace{(ty^4+t)}_{A(t,y)}dt + \underbrace{(2t^2y^3+y)}_{B(t,y)}dy = 0$$

(1) Is it exact?

$$\left.\begin{aligned} \frac{\partial A}{\partial y} &= 4ty^3 \\ \frac{\partial B}{\partial t} &= 4ty^3 \end{aligned}\right\}$$ agree! ODE is exact

(2) Solution

As ODE is exact there is $U(t,y) = C$

Integrate $A(t,y)$

$$U(t,y) = \int A(t,y)\,dt = \int ty^4 + t\,dt$$
$$= \frac{1}{2}t^2y^4 + \frac{1}{2}t^2 + F(y)$$

Integrate $B(t,y)$

$$\frac{\partial U(t,y)}{\partial y} = B(t,y)$$

Plug in temporary result for $U(x,y)$

$$2x^2y^3 + \frac{dF}{dy} = 2x^2y^3 + y$$

$$F = \int y\,dy = \frac{1}{2}y^2 + \underbrace{C_2}_{\text{redundant}}$$

Combine results for $U(x,y)$:

$$C = U(x,y) = \frac{1}{2}x^2y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + \underbrace{C_2}_{\text{redundant}}$$

$$\tilde{C} = x^2y^4 + x^2 + y^2 \quad \text{(implicit solution)}$$

this is a bi-quadratic equation:

set $y^2 = z$

$$x^2z^2 + z + x^2 - \tilde{C} = 0$$

$$z_{1/2} = \frac{-1 \pm \sqrt{1^2 - 4x^2(x^2 - \tilde{C})}}{2x^2} = y^2$$

$$y = \pm\sqrt{\frac{-1 \pm \sqrt{1 - 4x^2(x^2 - \tilde{C})}}{2x^2}}$$

c) $2x \frac{dy}{dx} + 3x + y = 0$

(0) Re-write ODE in standard form:

$$\underbrace{(3x+y)}_{A(x,y)} \cdot dx + \underbrace{2x}_{B(x,y)} \cdot dy = 0$$

(1) Check if ODE is exact:

$$\frac{\partial A}{\partial y} = 1$$

$$\frac{\partial B}{\partial x} = 2$$

do not agree!
ODE is not exact!

(2) Find integrating factor μ:

goal: find $\mu(x)$ s.t. it does not depend on y

New ODE:

$$\underbrace{\mu(x)(3x+y)}_{\text{new } A} dx + \underbrace{\mu(x) 2x}_{\text{new } B} \cdot dy = 0$$

We demand that the new ODE should be exact:

$$\frac{\partial A^{new}}{\partial y} = \frac{\partial B^{new}}{\partial x}$$

$$\mu(t)\cdot 1 = \frac{d\mu(t)}{dt}\cdot 2t + \mu(t)\cdot 2$$

$$(1-2)\frac{\mu(t)}{2t} = \frac{d\mu}{dt}$$ (separate and solve)

$$-\frac{dt}{2t} = \frac{d\mu}{\mu}$$

1

Since this calculation is always "the same"

$$\frac{d\mu}{\mu} = \frac{1}{B(t,y)}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial t}\right)dt$$

Solve ODE for $\mu(t)$:

$$\ln(\mu) = -\frac{1}{2}\ln(t) = \ln(t^{-1/2}) \quad | e^{\cdots}$$

$$\mu(t) = \frac{1}{\sqrt{t}}$$

New exact ODE:

$$\underbrace{\left(3\cdot\sqrt{t} + \frac{y}{\sqrt{t}}\right)}_{A\ \text{new}} dt + \underbrace{2\sqrt{t}}_{B\ \text{new}}\, dy = 0$$

↳ now solve using standard approach!