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Numerical Methods and Simulation
Module MT 03
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https://moodle.th-ab.de/course/view.php?id=624

Problem Set B

Seperable and exact ODEs

Homework

Exercise 1:

Solve the following ODEs:

(a) $y' \sin y + x = 0$ (b) $y' = -4x\sqrt{y-1}$ for $y \geq 1$ (c) $y'(1+x^2)x + 2y = 0$

Solution:

(a)

$$\begin{aligned}
y' \sin y + x &= 0 \\
\frac{dy}{dx} \cdot \sin y + x &= 0 \\
\frac{dy}{dx} &= -\frac{x}{\sin y} \\
\sin y \cdot dy &= -x \cdot dx \\
\int \sin y \cdot dy &= -\int x \cdot dx \\
-\cos y &= -\frac{1}{2}x^2 + C \\
\cos y &= \frac{1}{2}x^2 + C_1 \\
y(x) &= \arccos\left(\frac{1}{2}x^2 + C_1\right), \quad C_1 \in \mathbb{R}
\end{aligned}$$

(b)

$$\begin{aligned}
y' &= -4x\sqrt{y-1} \qquad y \geq 1 \\
\frac{dy}{dx} &= -4x\sqrt{y-1} \\
\frac{dy}{\sqrt{y-1}} &= -4x\, dx \\
\int \frac{dy}{\sqrt{y-1}} &= -\int 4x\, dx \\
2\sqrt{y-1} &= -2x^2 + C \\
\sqrt{y-1} &= -x^2 + C_1 \quad \text{da } \sqrt{y-1} \geq 0 \Rightarrow 0 \leq x^2 \leq C_1 \\
y &= 1 + (C_1 - x^2)^2
\end{aligned}$$

Deviding by $\sqrt{y-1}$ the additional constant solution $y = 1$ has been lost.

$$y(x) = \begin{cases} 1 + (C_1 - x^2)^2 & \text{für} \quad |x| \leq \sqrt{C_1} \\ 1 & \text{für} \quad |x| \geq \sqrt{C_1} \end{cases}$$

(c)

$$\begin{aligned}
y'(1+x^2)x + 2y &= 0\\
y' &= -\frac{2y}{x(1+x^2)}\\
\frac{dy}{dx} &= -\frac{2y}{x(1+x^2)}\\
\frac{dy}{2y} &= -\frac{dx}{x(1+x^2)}\\
\int \frac{dy}{2y} &= -\int \frac{dx}{x(1+x^2)} = -\int \left[\frac{1}{x} - \frac{x}{1+x^2}\right] dx\\
\frac{1}{2}\ln|y| &= -\ln|x| + \frac{1}{2}\ln(1+x^2) + C\\
\ln|y| &= -2\ln|x| + \ln(1+x^2) + C_1 = -\ln(x^2) + \ln(1+x^2) + C = \ln\frac{1+x^2}{x^2} + C\\
y &= \pm e^{C_1} \cdot \frac{1+x^2}{x^2} = C_2 \cdot \frac{1+x^2}{x^2}
\end{aligned}$$

Alternatively, use the integral

$$\int \frac{dx}{x(a^2+x^2)} = -\frac{1}{2a^2}\ln\left(\frac{a^2+x^2}{x^2}\right)$$

Exercise 2:

Solve the following ODEs by substitution:

(a) $y' = 4x - y$ (b) $xy' = y(\ln(y) - \ln(x))$ for $x, y > 0$ (c) $\frac{dy}{dx} = -\frac{x+y}{3x+3y-4}$

Solution:

(a)

$$\begin{aligned}
y' &= 4x - y\\
\text{Subst. } z &:= 4x - y \Rightarrow z' = 4 - y' \Rightarrow y' = 4 - z'\\
4 - z' &= z \Rightarrow \frac{dz}{dx} = 4 - z \Rightarrow \frac{dz}{4-z} = dx\\
\int \frac{dz}{4-z} &= \int dx \Rightarrow -\ln|4-z| = x + C\\
4 - z &= C_1 e^{-x}\\
z &= 4 - C_1 e^{-x}\\
z &= 4x - y \Rightarrow y = 4x - z = 4x - 4 + C_1 e^{-x}
\end{aligned}$$

(b)

$$\begin{aligned}
xy' &= y(\ln y - \ln x) = y\ln\left(\frac{y}{x}\right)\\
y' &= \frac{y}{x}\ln\left(\frac{y}{x}\right) \qquad \text{Subst. } z := \frac{y}{x} \quad y = zx\\
y' = z'x + z &= z\ln z \Rightarrow z'x = z(\ln z - 1) \Rightarrow \frac{dz}{z(\ln z - 1)} = \frac{dx}{x}\\
\int \frac{dz}{z(\ln z - 1)} &= \int \frac{\frac{1}{z}dz}{\ln z - 1} = \int \frac{dx}{x}\\
\ln(\ln z - 1) &= \ln x + C \quad \text{Zähler = Ableitung Nenner}\\
\ln z - 1 &= C_1 \cdot x\\
\ln z &= C_1 \cdot x + 1\\
z &= e^{C_1 x + 1}\\
y &= xz = xe^{C_1 x + 1}
\end{aligned}$$

(c) see sheet

Exercise 3:

Solve the following initial value problems:

(a) $y^2y' + x^2 = 1$ with $y(2) = 1$

(b) $x(x+1)y' = y$ with $y(1) = \frac{1}{2}$

Solution:

(a)

$$\begin{aligned}
y^2y' + x^2 &= 1 \Rightarrow y^2\,dy = (1-x^2)\,dx \Rightarrow \frac{1}{3}y^3 = x - \frac{1}{3}x^3 + C \\
y^3 &= 3x - x^3 + 3C \Rightarrow y = \sqrt[3]{3x - x^3 + 3C} \\
\text{IVP } y(2) &= \sqrt[3]{6 - 8 + 3C} = 1 \Rightarrow C = 1 \\
y &= \sqrt[3]{3x - x^3 + 3}
\end{aligned}$$

(b)

$$\begin{aligned}
x(x+1)y' &= y \Rightarrow \frac{dy}{y} = \frac{dx}{x(x+1)} = \frac{dx}{x} - \frac{dx}{x+1} \\
\ln|y| &= \ln|x| - \ln|x+1| + C \\
y &= e^{\ln|x| - \ln|x+1| + C} = e^C \cdot \frac{|x|}{|x+1|} \\
\Rightarrow y &= C_1\frac{x}{x+1} \\
\text{IVP } \ y(1) &= C_1 \cdot \frac{1}{2} = \frac{1}{2} \Rightarrow C_1 = 1 \\
y &= \frac{x}{x+1}
\end{aligned}$$

Exercise 4:

Check if the following ODEs are exact or can be made exact by an integrating factor. If possible solve the ODE.

(a) $(x^2+4xy)\,dx + (2x^2-y^2)\,dy = 0$ (b) $y(2x^2y^2+1)y'+x(y^4+1) = 0$ (c) $2xy'+3x+y = 0$

Solution:

(a)

$$\begin{aligned}
(x^2 + 4xy)dx + (2x^2 - y^2)dy &= 0 \\
P(x,y) = x^2 + 4xy &\Rightarrow \frac{\partial P}{\partial y} = 4x \\
Q(x,y) = 2x^2 - y^2 &\Rightarrow \frac{\partial Q}{\partial x} = 4x \\
&\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \\
u(x,y) &= \int P(x,y)dx = \int (x^2 + 4xy)dx = \frac{1}{3}x^3 + 2x^2y + u_1(y) \\
\frac{df}{dy} &= Q(x,y) = 2x^2 - y^2 = 2x^2 + \frac{\partial u_1}{\partial y} \\
&\Rightarrow \frac{\partial u_1}{\partial y} = -y^2 \Rightarrow u_1(y) = -\frac{1}{3}y^3 \\
\Rightarrow u(x,y) &= \frac{1}{3}x^3 + 2x^2y - \frac{1}{3}y^3 = C \iff x^3 + 6x^2y - y^3 = C_1
\end{aligned}$$