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Numerical Methods and Simulation
Module MT 03
Winter term 2021/22

https://moodle.th-ab.de/course/view.php?id=2907

Organizational Matters

Problem Set A

Calculation of Fourier coefficients

Homework

Exercise 1:

Consider the function $f(x) = x$ with $0 \leq x \leq 2\pi$ which is periodically continued on all $\mathbb{R}$.

a) Sketch the function

b) Represent the function by a Fourier series

c) Discuss the convergence of the Fourier series at the point $x = 0$.

d) Explain the values of the coefficients a_k.

Exercise 2:

Consider the 2π-periodic time signal

$$f(t) = \begin{cases} \cos(t) - \frac{1}{2} & 0 \leq |t| \leq \frac{\pi}{3} \\ 0 & \frac{\pi}{3} < |t| \leq \pi \end{cases}$$

a) Sketch the signal

b) Represent the function by a Fourier series

c) Calculate the first Fourier coefficients up to $k = 2$ explicitely.

d) Discuss the convergence of the Fourier series at the point $x = 0$.

Note: $\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$

Exercise 3:

Consider the 2π-periodic function

$$f(x) = \begin{cases} \sin|x| & 0 \leq |x| \leq \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < |x| \leq \pi \end{cases}$$

a) Sketch the function

b) Represent the function by a truncated Fourier series up to $k = 2$.

Note: $\cos(\alpha)\sin(\beta) = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))$

Exercise 4:

Consider the 2π-periodic time signal

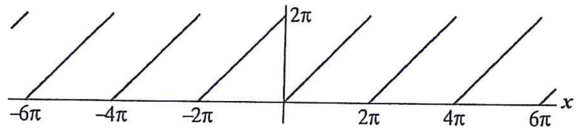
$$f(t) = \begin{cases} |t| & 0 \leq |t| \leq \frac{\pi}{2} \\ |t| + 1 & \frac{\pi}{2} < |t| \leq \pi \end{cases}$$

a) Sketch the signal

b) Represent the signal by its Fourier series and make the Fourier coefficients up to $k = 3$ explicit.

c) Against which value converges the Fourier series at $x = \pi/2$?

(1) a)



b) Periodenlänge 2π, $f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, \mathrm{d}x, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, \mathrm{d}x$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, \mathrm{d}x = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos kx \, \mathrm{d}x = \frac{1}{\pi} \int_{0}^{2\pi} x \cos kx \, \mathrm{d}x \quad \text{(wegen Periodizität)}$$

$$\int x \cos kx \, \mathrm{d}x = x \frac{\sin kx}{k} - \int \frac{\sin kx}{k} \mathrm{d}x = \frac{x \sin kx}{k} + \frac{\cos kx}{k^2}, \quad \text{falls } k \neq 0$$

$$k = 0: \quad a_0 = \frac{1}{\pi} \int_{0}^{2\pi} x \, \mathrm{d}x = \frac{1}{\pi} \frac{x^2}{2} \Big|_0^{2\pi} = 2\pi$$

$$k \neq 0: \quad a_k = \frac{1}{\pi} \int_{0}^{2\pi} x \cos kx \, \mathrm{d}x = \frac{1}{\pi} \left(\frac{x \sin kx}{k} + \frac{\cos kx}{k^2} \right) \Big|_0^{2\pi} = \frac{1}{\pi} \left(0 + \frac{1}{k^2} - 0 - \frac{1}{k^2} \right) = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, \mathrm{d}x = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin kx \, \mathrm{d}x = \frac{1}{\pi} \int_{0}^{2\pi} x \sin kx \, \mathrm{d}x \quad \text{(wegen Periodizität)}$$

$$\int x \sin kx \, \mathrm{d}x = x \left(-\frac{\cos kx}{k} \right) + \int \frac{\cos kx}{k} \mathrm{d}x = -\frac{x \cos kx}{k} + \frac{\sin kx}{k^2}$$

$k = 0:$ für Sinus–Koffizient nicht möglich.

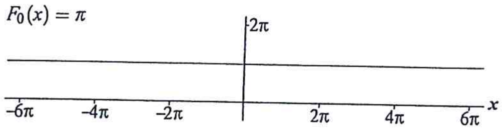
$$k \neq 0: \; b_k = \frac{1}{\pi} \int_{0}^{2\pi} x \sin kx \, \mathrm{d}x = \frac{1}{\pi} \left(-\frac{x \cos kx}{k} + \frac{\sin kx}{k^2} \right) \Big|_0^{2\pi} = \frac{1}{\pi} \left(-\frac{2\pi}{k} + 0 + 0 - 0 \right) = -\frac{2}{k}$$

Fourierentwicklung also

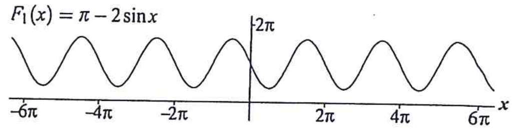
$$f(x) \sim \pi - \sum_{k=1}^{\infty} \frac{2}{k} \sin kx = \pi - 2 \sum_{k=1}^{\infty} \frac{\sin kx}{k} = \pi - 2 \left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots \right)$$

$$F_n(x) = \pi - 2 \sum_{k=1}^{n} \frac{\sin kx}{k}$$

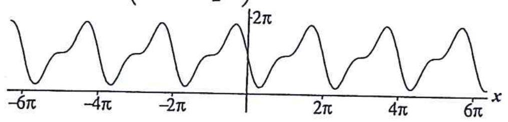
$F_0(x) = \pi$



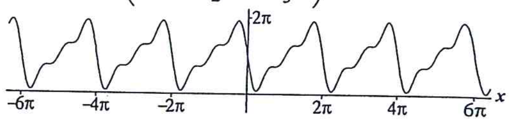
$F_1(x) = \pi - 2\sin x$



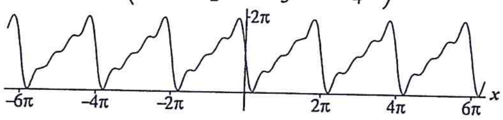
$F_2(x) = \pi - 2\left(\sin x + \frac{\sin 2x}{2}\right)$



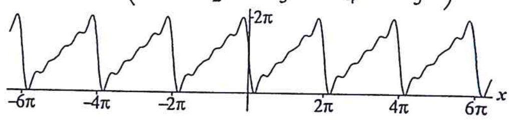
$F_3(x) = \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\right)$



$F_4(x) = \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4}\right)$



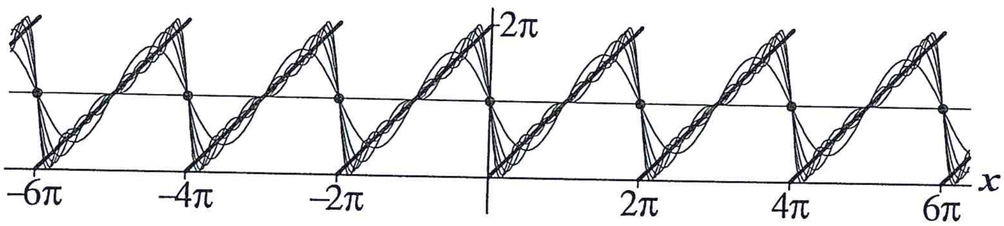
$F_5(x) = \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \frac{\sin 5x}{5}\right)$



c) Satz von Dirichlet: In Stetigkeitspunkten Konvergenz gegen $f(x)$,

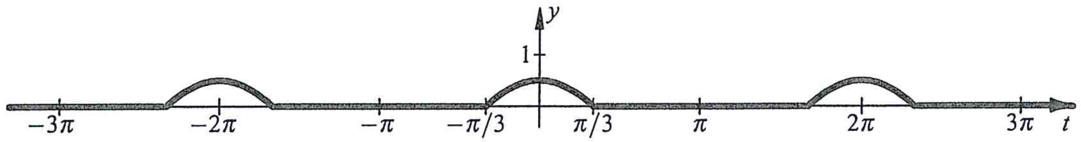
an Sprungstellen gegen $\frac{f(x+0)+f(x-0)}{2}$.

$$\text{Also: } F_n(x) \xrightarrow[n\to\infty]{} \begin{cases} f(x), & x \neq 2l\pi \\ \frac{2\pi+0}{2} = \pi, & x = 2l\pi \end{cases}.$$



d) $a_0 \neq 0$, $a_k = 0 \;\; \forall k > 0$ } shift of function $f(x) \to f(x) - \pi$ leads to antisymmetric function

2 a)



b) $f(t) \sim a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)$. Da $f(t)$ gerade ist, gilt $b_k = 0 \ \forall k$.

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\cos kt\,dt = \frac{2}{\pi}\int_0^{\pi/3}\left(\cos t - \frac{1}{2}\right)\cos kt\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi/3}\left(\cos t\,\cos kt - \frac{1}{2}\cos kt\right)dt = \frac{1}{\pi}\int_0^{\pi/3}(\cos(k+1)t + \cos(k-1)t - \cos kt)\,dt$$

$$= \underbrace{\frac{1}{\pi}\frac{\sin(k+1)t}{k+1}\Big|_0^{\pi/3}}_{=\frac{1}{\pi}\frac{\sin(k+1)\frac{\pi}{3}}{k+1}} + \underbrace{\frac{1}{\pi}\frac{\sin(k-1)t}{k-1}\Big|_0^{\pi/3}}_{=\frac{1}{\pi}\frac{\sin(k-1)\frac{\pi}{3}}{k-1}\ \text{für } k\neq 1} - \underbrace{\frac{1}{\pi}\frac{\sin kt}{k}\Big|_0^{\pi/3}}_{=\frac{1}{\pi}\frac{\sin k\frac{\pi}{3}}{k}\ \text{für } k\neq 0}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi/3}(2\cos t - 1)\,dt = \frac{1}{\pi}(2\sin t - t)\Big|_0^{\pi/3} = \frac{1}{\pi}\left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{\pi} - \frac{1}{3} \approx 0.217996$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi/3}(\cos 2t + 1 - \cos t)\,dt = \frac{1}{\pi}\left(\frac{\sin 2t}{2} + t - \sin t\right)\Big|_0^{\pi/3} = \frac{1}{\pi}\left(\frac{\sin\frac{2\pi}{3}}{2} + \frac{\pi}{3} - \sin\frac{\pi}{3}\right)$$

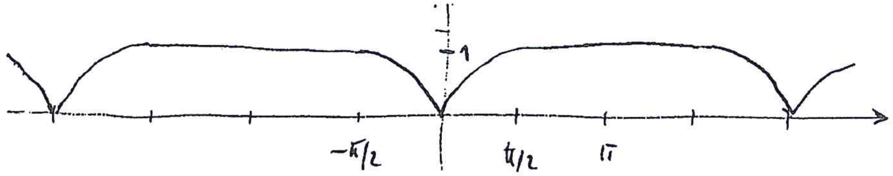
$$= \frac{1}{\pi}\left(\frac{1}{4}\sqrt{3} + \frac{\pi}{3} - \frac{1}{2}\sqrt{3}\right) = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \approx 0.195501$$

$$a_2 = \frac{1}{\pi}\frac{\sin\frac{3\pi}{3}}{3} + \frac{1}{\pi}\frac{\sin\frac{\pi}{3}}{1} - \frac{1}{\pi}\frac{\sin\frac{2\pi}{3}}{2} = \frac{1}{\pi}\frac{1}{2}\sqrt{3} - \frac{1}{\pi}\frac{1}{4}\sqrt{3} = \frac{\sqrt{3}}{4\pi} \approx 0.137832$$

$$f(t) \sim \frac{\sqrt{3}}{2\pi} - \frac{1}{6} + \left(\frac{1}{3} - \frac{\sqrt{3}}{4\pi}\right)\cos t + \frac{1}{\pi}\sum_{k=2}^{\infty}\left(\frac{\sin(k+1)\frac{\pi}{3}}{k+1} + \frac{\sin(k-1)\frac{\pi}{3}}{k-1} - \frac{\sin k\frac{\pi}{3}}{k}\right)\cos kt$$

$$\approx 0.108998 + 0.195501\cos t + 0.137832\cos 2t + \ldots$$

3 a)



b) $f(x)$ gerade $\Longrightarrow$ $b_k=0$ für alle k

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx=\frac{2}{\pi}\int_0^{\pi} f(x)\cos kx\,dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin x\cos kx\,dx+\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\cos kx\,dx$$

$$=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\Big(\sin(k+1)x-\sin(k-1)x\Big)\,dx+\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\cos kx\,dx$$

$$k=0:\quad a_0=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin x\,dx+\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}dx=-\frac{2}{\pi}\cos x\Big|_0^{\frac{\pi}{2}}+\frac{2}{\pi}\frac{\pi}{2}=\frac{2}{\pi}+1$$

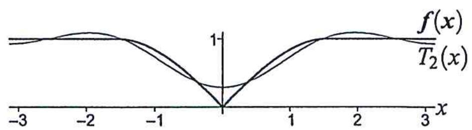
$$k=1:\quad a_1=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\sin 2x\,dx+\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\cos x\,dx=-\frac{\cos 2x}{2\pi}\Big|_0^{\frac{\pi}{2}}+\frac{2}{\pi}\sin x\Big|_{\frac{\pi}{2}}^{\pi}=-\frac{-1-1}{2\pi}+\frac{2}{\pi}(0-1)=\frac{1}{\pi}-\frac{2}{\pi}$$

$$=-\frac{1}{\pi}$$

$$k=2:\quad a_2=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}(\sin 3x-\sin x)\,dx+\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\cos 2x\,dx=-\frac{\cos 3x}{3\pi}\Big|_0^{\frac{\pi}{2}}+\frac{\cos x}{\pi}\Big|_0^{\frac{\pi}{2}}+\frac{\sin 2x}{\pi}\Big|_{\frac{\pi}{2}}^{\pi}=\frac{1}{3\pi}-\frac{1}{\pi}$$

$$=-\frac{2}{3\pi}$$

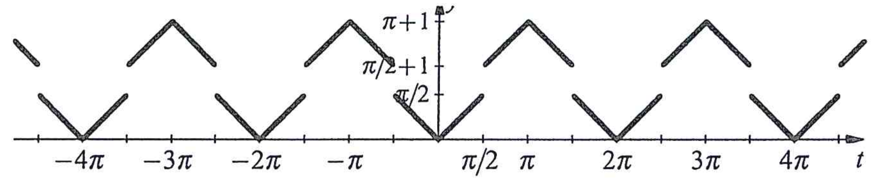
also $f(x)\approx T_2(x)=\frac{1}{\pi}+\frac{1}{2}-\frac{1}{\pi}\cos x-\frac{2}{3\pi}\cos 2x$



(Es ginge weiter mit $-\frac{1}{3\pi}\cos 3x-\frac{2}{15\pi}\cos 4x\ \ldots$.)

(4)

a)



b) $f(t)$ gerade, Periodenlänge 2π: $f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kt$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\cos kt\,dt = \frac{2}{\pi}\int_0^{\pi} t\cos kt\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi}\cos kt\,dt$$

$$\int t\cos kt\,dt = \frac{t\sin kt}{k} - \int\frac{\sin kt}{k}\,dt = \frac{t\sin kt}{k} + \frac{\cos kt}{k^2} + C \quad \text{für } k \neq 0$$

$$a_k = \left[\frac{2}{\pi}\left(\frac{t\sin kt}{k} + \frac{\cos kt}{k^2}\right)\right]_0^{\pi} + \left[\frac{2}{\pi}\frac{\sin kt}{k}\right]_{\pi/2}^{\pi} = \frac{2}{\pi}\left(\frac{\pi\sin k\pi - 0}{k} + \frac{\cos k\pi - 1}{k^2} + \frac{\sin k\pi - \sin\frac{k\pi}{2}}{k}\right)$$

$$= \frac{2}{\pi}\left(\frac{(-1)^k - 1}{k^2} - \frac{\sin\frac{k\pi}{2}}{k}\right) = \begin{cases} 0, & k \text{ gerade}, \neq 0 \\ \frac{2}{\pi}\left(-\frac{2}{k^2} - \frac{1}{k}\right) = -\frac{2k+4}{\pi k^2}, & k = 4l+1 \\ \frac{2}{\pi}\left(-\frac{2}{k^2} + \frac{1}{k}\right) = +\frac{2k-4}{\pi k^2}, & k = 4l+3 \end{cases}$$

In der Aufgabe ist nur verlangt: $a_1 = \frac{2}{\pi}(-2-1) = -\frac{6}{\pi}$, $a_2 = 0$, $a_3 = \frac{2}{\pi}\left(-\frac{2}{9} + \frac{1}{3}\right) = \frac{2}{9\pi}$.

$$a_0 = \frac{2}{\pi}\left(\int_0^{\pi} t\,dt + \int_{\pi/2}^{\pi} dt\right) = \frac{2}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi} + \frac{2}{\pi}\Big[t\Big]_{\pi/2}^{\pi} = \frac{2}{\pi}\left(\frac{\pi^2}{2} + \pi - \frac{\pi}{2}\right) = \frac{2}{\pi}\frac{\pi^2+\pi}{2} = \pi + 1$$

$$f(t) \sim \frac{\pi+1}{2} + \frac{1}{\pi}\left(-6\cos t + \frac{2}{9}\cos 3t - \frac{14}{25}\cos 5t + \frac{10}{49}\cos 7t \mp \cdots\right)$$

$$= \frac{\pi+1}{2} + \frac{1}{\pi}\sum_{l=0}^{\infty}\left(-\frac{2(4l+1)+4}{(4l+1)^2}\cos(4l+1)t + \frac{2(4l+3)-4}{(4l+3)^2}\cos(4l+3)t\right)$$

$$= \frac{\pi+1}{2} + \frac{1}{\pi}\sum_{l=0}^{\infty}\left(-\frac{8l+6}{(4l+1)^2}\cos(4l+1)t + \frac{8l+2}{(4l+3)^2}\cos(4l+3)t\right)$$

In der Aufgabenstellung ist nur verlangt: $f(t) =\sim \frac{\pi+1}{2} - \frac{6}{\pi}\cos t + \frac{2}{9\pi}\cos 3t + \cdots$.

Convergence to $\begin{cases} f(t) & t \neq (2l+1)\frac{\pi}{2} \\ \frac{\pi+1}{2} & t = (2l+1)\frac{\pi}{2} \end{cases}$