Kapitel 1

Fourier-Series

1.1 General remarks

1.1.1 Series representations of functions

Definition 1. *A set of functions* $\{b_i\}_i$ *is called a* basis *of a space of functions* [1] *definition* $\mathcal{F} = \{f | f : \mathbb{R} \supset D \to W \subset \mathbb{R}, x \to f(x)\}$ *if any function of that space can be represented as a* linear combination *of uniquely determined coefficients* $c_i \in \mathbb{R}$ *(combination with real coefficients) or* $c_i \in \mathbb{C}$ *(combination with complex coefficients) such that*

$$f(x) = \sum_i c_i \; b_i(x) \qquad \forall x \in D$$

The set of linear coefficients $\{c_i\}_i$ are 'coordinates' in function space. Equality only holds if the sum converges. This may require to restrict the space to some specific functions (continuous, differentiable, periodic, etc).

Definition 2 (Scalar product for vectors and functions)**.** *The* scalar product *(also called dot product, inner product) for two vectors* $\vec{u}, \vec{v} \in \mathbb{R}^n$ *is defined by the sum over the product of corresponding coefficients*

$$\vec{u} \cdot \vec{v} = \sum_i u_i v_i$$

In analogy, the scalar product of two functions $f, g \in \mathcal{F}$ *of a function space* $\mathcal{F}$ *is defined by the integral over the product of both functions*

$$\langle f | g \rangle = \int_D f(x) \; g(x) \; dx$$

Two functions f, g *are called* orthogonal *iff* $\langle f | g \rangle = 0$.
A basis of a function space is called orthogonal *iff* $\langle b_i | b_j \rangle = 0$ *for all* $i \neq j$ *and* orthonormal *iff, in addition,* $\langle b_i | b_i \rangle = 1$ *for all* i.

[1] Mathematically, a function space is a vector space. A full definition of a vector space is given in linear algebra.

Note: The scalar product can be understood as the projection of its two factors onto each other.

Example: Scalar products of important functions

1. $\int_0^{2\pi} \sin(kx)\sin(lx)\ dx = \begin{cases} 0 & \text{for} \quad k \neq l \\ \pi & \text{for} \quad k = l \neq 0 \\ 0 & \text{for} \quad k = l = 0 \end{cases}$

2. $\int_0^{2\pi} \cos(kx)\cos(lx)\ dx = \begin{cases} 0 & \text{for} \quad k \neq l \\ \pi & \text{for} \quad k = l \neq 0 \\ 2\pi & \text{for} \quad k = l = 0 \end{cases}$

3. $\int_0^{2\pi} \cos(kx)\sin(lx)\ dx = \{\ 0 \quad \text{for} \quad k, l \in \mathbb{R}$

1.1.2 Recap: Taylor expansion

Function space of (m+1)-fold differentiable functions

Let $\mathcal{I} \subset \mathcal{R}$ be an open interval, $\mathcal{F} = \{f | f : \mathbb{R} \supset I \to W \subset \mathbb{R}, x \to f(x)$, f (m+1) times differentiable in I $\}$ the function space of all in I $(m+1)$- times differentiable functions .

Taylor expansion

The Taylor expansion can be seen as a special form of an approximate series expansion of $f \in \mathcal{F}$ with respect to the basis of monomials $\{b_i\}_i = \{x^i\}_i$.

$$x \in \mathcal{I} \text{ is } f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x, x_0) \tag{1.1}$$

$$R_n(x, x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \text{ with } \min(x, x_0) < \xi < \max(x, x_0)$$

The crucial question is for which values $x \in I$ the Taylor series converges.

Example: Taylor series of the exponential function

It holds $\forall n : f^{(n)}(x) = \exp(x)$ and $\forall n : f^{(n)}(0) = 1$. Then

$$T(x, x_0 = 0) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n \quad \text{(Taylor series of exponential function)} \tag{1.2}$$

1.1.3 Periodic functions

Definition 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. f is called* periodic *or* T-periodic *with a period $T \in \mathbb{R}^+$ if*

$$f(t+T) = f(t) \quad \forall t \in \mathbb{R}$$

For $T = 2\pi$ we call the function 2π periodic.

(1) $\sin(kx), \cos(kx)$ $k \in \mathbb{Z}$ are $2\pi/k-$ periodic (higher frequencies)

(2) $f(x) = \begin{cases} 1 & \text{for} \quad x \in [2k, 2k+1] \\ 0 & \text{for} \quad x \in [2k+1, 2k+2[\end{cases}$ $k \in \mathbb{Z}$, is 2 periodic

(3) $f(x) = \begin{cases} (x-2k) & \text{for} \quad x \in [2k, 2k+1] \\ 1-(x-(2k+1)) & \text{for} \quad x \in [2k+1, 2k+2[\end{cases}$ $k \in \mathbb{Z}$

(4) $f(x) = x$ in $[-\pi, \pi]$ (2π-periodic sawtooth function)

Theorem 1 (Translation of domain)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a T-periodic function and $x_0 \in \mathbb{R}$. Then*

$$\int_0^T f(x)\,dx \;=\; \int_{x_0}^{x_0+T} f(x)\,dx \quad \forall x \in \mathbb{R}$$

i.e. integrals over a full period of a periodic function can be arbitrarily shifted.

1.2 Fourier series

1.2.1 Fourier basis

Definition 4 (Fourier basis)**.** *Consider an interval $[x_0, x_0+T]$ of length $T \in \mathbb{R}$.*

- *The set of T-periodic real functions $\{\cos(2\pi kx/T), \sin(2\pi kx/T)\}_{k\in\mathbb{N}}$ are called real Fourier basis functions.*
- *The set of T-periodic complex functions $\{e^{i(2\pi kx/T)} = \cos(2\pi kx/T) + i\sin(2\pi kx/T)\}_{k\in\mathbb{Z}}$ are called complex Fourier basis functions.*

1.2.2 Scalar products of the real Fourier basis functions

Consider an interval $D = [x_0, x_0+T]$ where $T \in \mathbb{R}$.

1. $\int_{x_0}^{x_0+T} \sin\left(\frac{2\pi kx}{T}\right) \sin\left(\frac{2\pi lx}{T}\right)\,dx = \begin{cases} 0 & \text{for} \quad k \neq l \\ T/2 & \text{for} \quad k = l \neq 0 \\ 0 & \text{for} \quad k = l = 0 \end{cases}$

2. $\int_{x_0}^{x_0+T} \cos\left(\frac{2\pi kx}{T}\right) \cos\left(\frac{2\pi lx}{T}\right)\ dx = \begin{cases} 0 & \text{for} \quad k \neq l \\ T/2 & \text{for} \quad k = l \neq 0 \\ T & \text{for} \quad k = l = 0 \end{cases}$

3. $\int_{x_0}^{x_0+T} \cos\left(\frac{2\pi kx}{T}\right) \sin\left(\frac{2\pi lx}{T}\right)\ dx = \ 0 \quad \text{for} \quad k, l \in \mathbb{R}$

1.2.3 Fourier series representation

Definition 5. *The* Fourier series representation *of a T-periodic function f which complies with the Dirichlet conditions (see below) is given by a linear superposition of the respective Fourier basis functions with real coefficients $a_k, b_k \in \mathbb{R}$ (real Fourier series) or complex coefficients $c_k \in \mathbb{C}$ (complex Fourier series)*

$$S_f(x) = \begin{cases} \frac{a_0}{2} + \sum_k a_k \cos\left(\frac{2\pi kx}{T}\right) + b_k \sin\left(\frac{2\pi kx}{T}\right) & \text{for} \quad k \in \mathbb{N} \text{ (real Fourier series)} \\ \sum_k c_k e^{\frac{2\pi i}{L}kx} & \text{for} \quad k \in \mathbb{Z} \text{ (complex Fourier series)} \end{cases}$$

The coefficients a_k, b_k (or c_k) are called Fourier coefficients *and constitute the real (or complex) Fourier series.*
As $1/T =: f_0$ is a frequency, we can say that the (Hertzian) frequencies involved in the Fourier series representation are given by $f_k = k \cdot f_0$.
Alternatively, the (angular) frequencies $\omega_0 = 2\pi/T$ and $\omega_k = k \cdot \omega_0$ are used.

Remark: The Fourier series has generally an infinite number of nonzero elements. However, for all relevant functions the Fourier coefficients decay with some power of $1/k$.

Theorem 2 (Dirichlet conditions)**.** *A function $f : \mathbb{R} \supset D \to W \subset \mathbb{R}$ can be expanded as a Fourier series if*

1. *f is periodic*
2. *f is continuous almost everywhere; a finite number of finite discontinuities (jumps) is however possible.*
3. *has only a finite number of minima and maxima within one period, i.e. it is not oscillating too rapidly*
4. *The integral over one period $\int_T |f(x)|dx < \infty$ must converge.*

Theorem 3. *The expansion of a function for which the Dirichlet conditions hold into a Fourier series is unique.*

1.3 Calculation of Fourier coefficients

Theorem 4 (Fourier coefficients). *Let f be a T-periodic function such that the Dirichlet conditions hold. The Fourier coefficients of f are given by*

$$a_k = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\ \cos\left(\frac{2\pi kx}{T}\right) dx$$
$$b_k = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\ \sin\left(\frac{2\pi kx}{T}\right) dx$$

If a function is periodic it remains periodic under any shift of the interval $[x_0, x_0+T[$. It is helpful to choose the simplest (most symmetric) interval.

Theorem 5 (Symmetry properties of Fourier coefficients). *1. A function $f(x) = f(-x)\ \forall x$ is called* symmetric w.r.t. the y axis*. Then the real Fourier series contains no* sin */ terms and all b_k coefficients vanish.*

2. A function $f(x) = -f(-x)\ \forall x$ is called antisymmetric around the origin*. Then the real Fourier series contains no* cos */ terms and all a_k coefficients vanish.*

1.3.1 Examples

1. The simplest function is the constant function $f(x) = 1\ \forall x$. It is periodic for any period T and symmetric on intervalls of the form $[-T/2, T/2[$. Hence $b_k = 0\ \forall k$.

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} 1\cdot 1\ dx = 2$$
$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} \cos\left(\frac{2\pi kx}{T}\right) dx = \left[\frac{1}{\pi k}\sin\left(\frac{2\pi kx}{T}\right)\right]_{-T/2}^{T/2} = 0 \quad \forall k$$

   Obviously, the Fourier representation $S_f(x) = a_0/2 + 0 = 1$ is trivial. This is because the constant function is an element of the Fourier basis (for k=0).

2. Consider the function $f(x) = \begin{cases} -1 & \text{for} \quad x \in [-1/2\ T, 0[\\ 1 & \text{for} \quad x \in [0, 1/2\ T[\end{cases}$
   Note that this function is odd. Therefore, all cosine integrals vanish. Thus $a_k = 0\ \forall k \in \mathbb{N}$. Note that $x_0 = -T/2$.

$$\begin{aligned}
b_k &= \frac{2}{T}\int_{-T/2}^{T/2} f(x)\ \sin\left(\frac{2\pi kx}{T}\right) dx \\
&= \frac{2}{T}\int_{-T/2}^{0} (-1)\ \sin\left(\frac{2\pi kx}{T}\right) dx + \frac{2}{T}\int_{0}^{T/2} \sin\left(\frac{2\pi kx}{T}\right) dx \\
&= 2\ \frac{2}{T}\int_{0}^{T/2} \sin\left(\frac{2\pi kx}{T}\right) dx \qquad \text{(special case because of symmetry)} \\
&= 2\ \frac{1}{\pi k}\int_{0}^{T/2} \sin\left(\frac{2\pi kx}{T}\right) d\left(\frac{2\pi kx}{T}\right) \qquad \text{(substitution)} \\
&= 2\ \frac{1}{\pi k}\int_{0}^{\pi k} \sin(y)\, dy \\
&= 2\ \frac{1}{\pi k}[-\cos(y)]_0^{k\pi} = \frac{2}{\pi k}[1-(-1)^k] = \begin{cases} 0 & \text{for } k \text{ even} \\ 4/(\pi k) & \text{for } k \text{ odd} \end{cases}
\end{aligned}$$

3. Other examples: Fourier series representations for periodic functions in $[-\pi, \pi]$.

function	*Fourier series*
x	$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$
$\lvert x\rvert$	$\frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos nx$
x^2	$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$
x^3	$2\sum_{n=1}^{\infty}(-1)^n \left(\frac{6}{n^3} - \frac{\pi^2}{n}\right) \sin nx$

1.3.2 Convergence of the Fourier series and Gibbs phenomenon

If the Dirichlet conditions hold the Fourier series $S_f(x)$ converges towards the function $f(x)$ in the following way:

$$S_f(x) = \frac{1}{2}\left(f(x+) + f(x-)\right) = \frac{1}{2}\lim_{\epsilon\to 0}\left(f(x+\epsilon) + f(x-\epsilon)\right)$$

(1) If the function is n times continuous differentiable on the periodic interval the Fourier series converges quickly towards the function. The Fourier coefficients decay with $1/k^n$.

(2) If the function is discontinuous at a finite number of points within its period the Fourier series converges towards the function at all points except at the discontinuities. At an discontinuity x_0, the Fourier series converges to an intermediate point:

$$\lim_{n\to\infty} S_f(x_0) \to \frac{1}{2}\left(f(x_0+) + f(x_0-)\right)$$

In this case the Fourier coefficients decay only like $1/k$.

Gibbs phenomenon: In a close environment around the discontinuity the Fourier series overshoots even in the limit of an infinite number of terms (Gibbs phenomenon). The overshooting is by roughly 9 percent of the function step at the dicontinuity. The relative size of the overshoot directly at the discontinuity cannot be influenced significantly by adding more Fourier coefficents to the Fourier series representation of the function.

1.3.3 Parseval's theorem

Theorem 6. *Let f be a function such that the Dirichlet conditions hold. Then*

$$\frac{1}{T}\int_{x_0}^{x_0+T} |f(x)|^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right)$$

Example: Power dissipation in an Ohmic resistor

Consider a AC voltage $V(t) = V_0 \sin(\omega t)$ applied to an Ohmic resistor. Obviously, the only nonzero Fourier coefficient is $b_1 = V_0$. The power dissipated at an resistor is related to the voltage by $P = V \cdot I = V^2/R$. Over a period, the power $P = \int_T V^2(t)dt/R$ is dissipated. Using Parseval's theorem we can calculate the dissipated power directly from the Fourier coefficients $P = (1/2)\ b_1^2 = (1/2)\ V_0^2$.

1.4 Alternative representations of the Fourier series

1.4.1 Amplitude and phase representation

In the (generic) Fourier series of a T-periodic function $f(x)$ both sin and cos functions appear pairwise with the same frequency. Those terms can be combined:

$$a_k \cos\left(\frac{2\pi}{T}kx\right) + b_k \sin\left(\frac{2\pi}{T}kx\right) = A_k \cos\left(\frac{2\pi}{T}kx + \varphi_k\right)$$

The new coefficients (A_k) and (φ_k) are called the amplitude spectrum and the phase spectrum of the function $f(x)$. They are related to the Fourier coefficients a_k and b_k by

$$A_k = \sqrt{a_k^2 + b_k^2} \qquad \text{and} \qquad \varphi_k = -\arctan\left(\frac{b_k}{a_k}\right) + \begin{cases} 0 & a_k > 0 \\ \pi & a_k < 0 \end{cases}$$

In an amplitude and phase picture the Fourier series reads

$$S_f(x) = a_0 + \sum_{k=1}^{\infty} A_k \cos\left(\frac{2\pi}{T} kx + \varphi_k\right)$$

1.4.2 Complex representation

We use the Euler relation for defining the complex exponential function

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$$

As the sin is antisymmetric while the cos is a symmetric function this implies

$$e^{-i\alpha} = \cos(\alpha) - i\sin(\alpha)$$

Adding / subtracting both equations gives representations of the trigonometric functions in terms of the complex exponential function:

$$\cos(\alpha) = \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) \qquad \text{and} \qquad \sin(\alpha) = \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right)$$

Inserting this representation into the Fourier series $S_f(x)$ allows to re-write it as a complex series:

$$\begin{aligned} S_f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi}{T} kx\right) + b_k \sin\left(\frac{2\pi}{T} kx\right) \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{a_k}{2}\left(e^{i\frac{2\pi}{T}kx} + e^{-i\frac{2\pi}{T}kx}\right) + \frac{b_k}{2i}\left(e^{i\frac{2\pi}{T}kx} - e^{-i\frac{2\pi}{T}kx}\right) \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{a_k - ib_k}{2} e^{i\frac{2\pi}{T}kx} + \frac{a_k + ib_k}{2} e^{-i\frac{2\pi}{T}kx} \\ &= \sum_{k=-\infty}^{\infty} c_k \; e^{i\frac{2\pi}{T}kx} \end{aligned}$$

The coefficients c_k follow by comparison.

The Fourier coefficients of the complex Fourier series can be calculated directly using

$$c_k = \frac{1}{T}\int_0^T f(x)\; e^{i\frac{2\pi}{T}kx}\, dx \qquad \forall k \in \mathbb{Z}$$

Note that the complex Fourier coefficients are related by complex conjugation $c_{-k} = c_k^*$. Hence, only half the number needs to be calculated!

Vice versa holds: $a_k = 2\ \mathrm{Re}(c_k)$ and $b_k = -2\ \mathrm{Im}(c_k)$.

1.4.3 Parseval's theorem

Theorem 7. *Let f be a function such that the Dirichlet conditions hold. Then*

$$\frac{1}{T}\int_{x_0}^{x_0+T} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |c_k|^2$$

1.5 Remark: (Continuous) Fourier transform

For nonperiodic functions $g(x)$ a similar Fourier analysis is (sometimes) possible. Then, $g(x)$ is represented by an integral over trigonometric or (complex exponential) functions known as the (real or complex) Fourier transform:

$$\begin{aligned} g(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{g}(k)e^{ikx}dk \\ \hat{g}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-ikx}dx \end{aligned}$$

The continuous Fourier transform is an integral transform which has a forward ($g(x) \to \hat{g}(k)$) and a backward ($\hat{g}(k) \to g(x)$) direction. The function $\hat{g}(k)$ is called the (continuous) Fourier transform of $g(x)$ and is a continuous generalization of the discrete complex Fourier coefficients c_k.

The commonly used prefactor (here $1/\sqrt{2\pi}$) differs between different fields (physics, signal processing, etc.). The product of the prefactors of the forward and backward transformation must be $1/(2\pi)$. Here, it is symmetrically attributed to both directions to ensure that Parseval's theorem remains unchanged.

Inserting the forward transform into the backward transform gives

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x')e^{-ikx'}dx'e^{ikx}dk$$

It is a deep mathematical proof to show the identity of both sides for suitable functions. Unfortunately, it does not hold for the simplest functions like $g(x) = 1$ because their integral over all space diverges. Hence, one important constraint on Fourier transforms is that both the function and its Fourier transform must be adequately integrable.

1.6 Real discrete Fourier series/transform (rDFT)

In numerical calculations, a function is known only on a finite number of discrete points $(x_i, f(x_i))$. We assume that all points are equally spaced $h = x_i - x_{i-1} \forall i$.

A discrete Fourier series of order N of a T-periodic discrete function f contains $2N$ parameters (the a_k and b_k coefficients). In order to determine

$2N$ parameters $2N$ independent points are needed within the period, i.e. within interval $[0, T]$. Hence, the stepsize of the discrete sampling of a continuous function must be $\Delta x = T/(2N)$. Periodicity implies that $f(0) = f(T)$, i.e. those points are not independent.

Note that there are modifications to the Fourier series due to sampling. One still writes the Fourier series of a discete function

$$S_f^D(x) = \sum_{k=0}^{N} \left[A_k \cos\left(\frac{2\pi kx}{T}\right) + B_k \sin\left(\frac{2\pi kx}{T}\right) \right]$$

However, the discrete Fourier coefficients change w.r.t. the a_k, b_k:

$$\begin{aligned}
A_0 &= \frac{1}{2N} \sum_{j=1}^{2N} f(x_j) \qquad \text{(mean value)} \\
A_k &= \frac{1}{N} \sum_{j=1}^{2N} f(x_j) \cos\left(\frac{2\pi k x_j}{T}\right) \qquad 1 \le k \le N-1 \\
A_N &= \frac{1}{2N} \sum_{j=1}^{2N} f(x_j) \cos\left(\frac{2\pi N x_j}{T}\right) \\
B_0 &= 0 \\
B_k &= \frac{1}{N} \sum_{j=1}^{2N} f(x_j) \sin\left(\frac{2\pi k x_j}{T}\right) \qquad 1 \le k \le N-1 \\
B_N &= 0
\end{aligned}$$

These results follow from the orthogonality relations for discrete scalar products and make use of the geometric series. Details of the derivation can be found in Amos Gilat, Vish Subramaniam, *Numerical Methods for Engineers and Scientists*, Wiley, (Appendix C).

Comparing the real discrete Fourier series and the (infinite) Fourier series of a continuous function

A continuous T-periodic function can be represented as a Fourier series with an infinite number of Fourier coefficients.

$$S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi kx}{T}\right) + b_k \sin\left(\frac{2\pi kx}{T}\right)$$

We can expect that $S_f(x_i) = S_f^D(x_i)$ for all $0 \le i \le 2N$. Equating both expressions on the 2N points $x_i = i \cdot T/(2N)$ shows that

$$A_k = a_k + \sum_{m=1}^{\infty} (a_{2N\cdot m-k} + a_{2N\cdot m-k}), \quad A_0 = a_0 + \sum_{m=1}^{\infty} a_{2N\cdot m},$$

$$B_k = b_k + \sum_{m=1}^{\infty}(-b_{2N\cdot m-k} + b_{2N\cdot m-k})$$

i.e. every coefficient of the discrete Fourier series contains an infinite summation of the coefficients of the infinite continuous Fourier series.

1.7 Aliasing and Shannon-Nyquist theorem

The addition of higher Fourier coef ficients due to finite frequency sampling gives rise to the phenomenon of *aliasing*.

Example: Anticlockwise movement appears clockwise

Consider an additional hand of a clock which runs **anticlockwise** with a rate of one revolution per minute (so its frequency is 1 per minute). Consider two measurements which *uniformly* sample the movement:

(1) The first measurement is done with a frequency of 4 per minute:

Time [sec]	0	15	30	45	60
Position of extra hand on clock	12	9	6	3	12

(2) The second measurement is done with a frequency of 4/3 per minute:

Time [min:sec]	0	45	1:30	2:15	3
Position of extra hand on clock	12	3	6	9	12

The second measurement looks as if it runs **clockwise** with a frequency of 1/3 per minute. The observed frequency is the difference between the actual frequency (1 per minute) and the measurement frequency (4/3 per minute).

Theorem 8 (Nyquist-Shannon)**.** *A bandwith-limited analog signal with a maximum frequency f_H [Hz] can be reconstructed without distortion from uniform samples if the sampling has been done with a sampling frequency f_s at least twice as large as the maximum frequency*

$$f_s \geq 2f_H = f_{\text{Nyquist}}$$

This lower threshold frequency is called the Nyquist frequency.

When the sampling frequency is lower than the Nyquist frequencz the signal is called *undersampled*.

Kapitel 2

Regression

The concept of a series representation of functions can be used in a more general way to approximate functions by other functions and to fit functions to a set of data points.

In regression problems, a function is represented by a finite linear combination of basis functions $b_i(x)$. The linear coefficients a_i can be calculated from fitting the function optimally to a set of n data points (x_i, y_i).

$$f(x) = \sum_{i=1}^{m} a_i \; b_i(x)$$

Evaluated for n data points this leads to a system of coupled linear equations which can be represented by a matrix equation

$$A\vec{x} = \vec{b} \qquad \text{with} \quad A \in \mathbb{R}^{n\times m}$$

The fitting of data points is an approximate procedure which is based on an optimization strategy. In the simplest case the parameters are chosen such that the squared error function

$$E = \sum_{i=1}^{n} \left[y_i - f(x_i; a_1 \ldots a_n)\right]^2$$

is minimized. Then the optimal parameters for the linear combination of basis functions is found by solving the system of linear equations using the Gaussian approximation

$$A^T A\vec{x} = A^T\vec{b}$$

Example