Problem Set B, Ex 5:

$$\frac{dy}{dx} + \underbrace{\frac{x}{a^2+x^2}}_{p(x)} \cdot \underset{\substack{\uparrow \\ \text{linear}}}{y} = \underbrace{x}_{\substack{q(x)\ (\text{perturbation}) \\ \text{inhomogeneous}}}$$

1) General solution of homogeneous ODE: set $q(x) = 0$

$$\frac{dy}{dx} = -\underbrace{\frac{x}{a^2+x^2}}_{g(x)} \cdot \underbrace{y}_{h(y)}$$

$\rightarrow$ seperable

$\Rightarrow$ seperation of variables

$$\int_{y_0}^{y} \frac{dy}{y} = -\int_{x_0}^{x} \frac{x}{a^2+x^2}\, dx = -\frac{1}{2}\int_{x_0}^{x} \frac{2x}{a^2+x^2}\, dx$$

Note: $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$

Here: $f(x) = a^2+x^2$, $f'(x) = 2x$

$$\ln y/y_0 = -1/2 \left[\ln|a^2+x^2|\right]_{x_0}^{x} = \ln\left(\frac{a^2+x^2}{a^2+x_0^2}\right)^{-1/2}$$

exponentiate to get $y_h(x)$ (homogeneous solution)

$$y_{hom}(x) = y_0 \cdot \sqrt{\frac{a^2+x_0^2}{a^2+x^2}}$$

Test of general solution:

$$y_{hom} = y_0 \cdot \left(\frac{\alpha^2 + x^2}{\alpha^2 + x_0^2} \right)^{-1/2}$$

$$y'_{hom} = -\frac{1}{2} y_0 \cdot \left(\frac{\alpha^2 + x^2}{\alpha^2 + x_0^2} \right)^{-3/2} \cdot \underbrace{\frac{2x}{\alpha^2 + x_0^2}}_{\text{chain rule}}$$

insert both into ODE:

$$-\frac{x y_0}{\alpha^2 + x_0^2} \left(\frac{\alpha^2 + x^2}{\alpha^2 + x_0^2} \right)^{-3/2} \overset{?}{=}$$

$$-\frac{x}{\alpha^2 + x^2} \cdot y_0 \cdot \left(\frac{\alpha^2 + x^2}{\alpha^2 + x_0^2} \right)^{-1/2}$$

$$-\underbrace{\frac{x y_0}{\cancel{\alpha^2 + x_0^2}} \cdot \left(\frac{\cancel{\alpha^2 + x_0^2}}{\alpha^2 + x^2} \right)} \cdot \left(\frac{\alpha^2 + x^2}{\alpha^2 + x_0^2} \right)^{-1/2} =$$

$$= -\frac{x}{\alpha^2 + x^2} y_0 \cdot \left(\frac{\alpha^2 + x^2}{\alpha^2 + x_0^2} \right)^{-1/2}$$

ok ✓

2) Particular solution of inhomogeneous ODE:

2.1 Integrating factor

$$\mu(x) = e^{\int p(x)\,dx} = e^{\frac{1}{2}\int \frac{2x}{a^2+x^2}\,dx}$$

$$= e^{\ln(a^2+x^2)^{1/2}} = \sqrt{a^2+x^2}$$

2.2. Particular solution:

$$y_{part} = \frac{1}{\mu}\cdot\int_{x_0}^{x} \mu(x)\, q(x)\, dx =$$

$$= \frac{1}{\sqrt{a^2+x^2}} \int_{x_0}^{x} \sqrt{a^2+x^2}\cdot x\cdot 2\cdot \frac{1}{2}\, dx$$

$$= \frac{1}{\sqrt{a^2+x^2}} \int_{x_0}^{x} \underbrace{\sqrt{a^2+x^2}\cdot(2x)\cdot\frac{3}{2}}_{\text{derivative of } (a^2+x^2)^{3/2}}\cdot\frac{2}{3}\cdot\frac{1}{2}\,dx$$

$$= \frac{1}{3}\,\frac{1}{\sqrt{a^2+x^2}}\left[(a^2+x^2)^{3/2}\right]_{x_0}^{x}$$

$$= \frac{1}{3}\left[(a^2+x^2) + (a^2+x_0^2)\sqrt{\frac{a^2+x_0^2}{a^2+x^2}}\right]$$

2.3. Full solution

$$y = y_{hom} + y_{part} = \frac{1}{3}(a^2+x^2) + C\sqrt{\frac{a^2+x_0^2}{a^2+x^2}}$$